Haverford Problem Solving Group
April 4, 2024
Miscellaneous Problems

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1. Let F_n denote the n-th Fibonacci number. Prove that 3^{2023} divides

$$3^2 \cdot F_4 + 3^3 \cdot F_6 + 3^4 \cdot F_8 + \cdots + 3^{2023} F_{4046}.$$

[ICMC Nov 2023/1]

2. Prove that there exist distinct positive integers $a_1, a_2, \ldots, a_{2024}$ such that, for each $i \in \{1, 2, \ldots, 2024\}$,

$$a_i \mid a_1 a_2 \cdots a_{i-1} a_{i+1} \cdots a_{2024} + k,$$

where (a) $k = 1$ and (b) $k = 2024$. [ICMC Feb 2024/1]

3. Fredy starts at the origin of the Euclidean plane. Each minute, Fredy may jump a positive integer distance to another lattice point, provided the jump is not parallel to either axis. Can Fredy reach any given lattice point in 2023 jumps or less? [ICMC Nov 2023/2]

4. Let $n \geq 3$ be a positive integer. A circular necklace is called fun if it has n black beads and n white beads. A move consists of cutting out a segment of consecutive beads and reattaching it in reverse. Prove that it is possible to change any fun necklace into any other fun necklace using at most $(n-1)$ moves. (Necklaces related by rotations or reflections are considered to be the same.) [ICMC Feb 2024/2]

5. There are 105 users on the social media platform Mathsenger, every pair of which has a direct messaging channel. Prove that each messaging channel may be assigned one of 100 encryption keys, such that no 4 users have the 6 pairwise channels between them all being assigned the same encryption key. [ICMC Nov 2023/3]

6. Let $(t_n)_{n \geq 1}$ be the sequence defined by $t_1 = 1, t_{2k} = -t_k$ and $t_{2k+1} = t_{k+1}$ for all $k \geq 1$. Consider the series

$$\sum_{n=1}^{\infty} \frac{t_n}{n^{1/2024}}.$$

Prove that this series converges to a positive real number. [ICMC Feb 2024/4]

For more problems, turn the page.

7. Points A, B, C, and D lie on the surface of a sphere with diameter 1. Determine the maximum possible volume of tetrahedron $ABCD$. [ICMC Nov 2023/4]

8. (a) Is there a non-linear integer-coefficient polynomial $P(x)$ and an integer N such that all integers greater than N may be written as the greatest common divisor of $P(a)$ and $P(b)$ for positive integers $a > b$?

(b) Is there a non-linear integer-coefficient polynomial $Q(x)$ and an integer M such that all integers greater than M may be written as $Q(a) - Q(b)$ for positive integers $a > b$?

[ICMC Nov 2023/5]

9. Is it possible to dissect an equilateral triangle into three congruent polygonal pieces (not necessarily convex), one of which contains the triangle's centre in its interior? (The interior of a polygon is the polygon without its boundary.) [ICMC Feb 2024/5]

10. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection of the positive integers. Prove that at least one of the following limits is true:

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n + f(n)} = \infty; \qquad \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{f(n)}\right) = \infty.$$

[ICMC Nov 2023/6]

11. A town has n residents, and they are members of some clubs (residents can be members of more than one club). No matter how we choose some (but at least one) clubs, there is a resident of the town who is the member of an odd number of the chosen clubs. Prove that the number of clubs is at most n. [KöMaL/A842]

12. Let N be the set of those positive integers n for which $n \mid k^k - 1$ implies $n \mid k - 1$ for every positive integer k. Prove that if $n_1, n_2 \in N$, then their greatest common divisor is also in N.

[KöMaL/A843]

13. For real number r let $f(r)$ denote the integer that is the closest to r (if the fractional part of r is $1/2$, let $f(r)$ be $r - 1/2$). Let $a > b > c$ rational numbers such that for all integers n the following is true: $f(na) + f(nb) + f(nc) = n$. Determine the possible values of a, b and c.

[KöMaL/A849]