Haverford Problem Solving Group
February 22, 2024
Miscellaneous Problems

0. Sign up **now** to join other members of the PSG at the Triceratops competition, **next Saturday,** 9am–2pm, at Swarthmore College.

1. Evaluate the sum

$$11^2 - 1^2 + 12^2 - 2^2 + 13^2 - 3^2 + \ldots + 20^2 - 10^2. \quad [1]$$

2. Find all ordered pairs of real numbers (x, y) such that $x^2y = 3$ and $x + xy = 4$. [2]

3. There are 15 students in PSG. After each meeting, 3 students stay in the Math Lounge to solve problems together. At the end of the year, it was discovered that each pair of students had been together in the Math Lounge exactly once. How many meetings did PSG have?

4. Given that $a+b+c = 5$ and that $1 \leq a, b, c \leq 2$, determine the the minimum possible value of

$$\frac{1}{a+b} + \frac{1}{b+c}. \quad [1]$$

5. Determine the remainder of the division by 7 of the sum

$$2^{\frac{1\cdot 2}{2}} + 2^{\frac{2\cdot 3}{2}} + \cdots + 2^{\frac{2011\cdot 2012}{2}}. \quad [2]$$

6. Find all real values of x for which

$$\frac{1}{\sqrt{x} + \sqrt{x-2}} + \frac{1}{\sqrt{x+2} + \sqrt{x}} = \frac{1}{4}. \quad [2]$$

7. Determine the period of the function $f(x) = \cos(\cos(x))$. [1]

8. Let ABC be a triangle, and let D, E, and F be the midpoints of sides BC, CA, and AB, respectively. Let the angle bisectors of $\angle FDE$ and $\angle FBD$ meet at P. Given that $\angle BAC = 37°$ and $\angle CBA = 85°$, determine the degree measure of $\angle BPD$. [2]

For more problems, turn the page.

9. How many subsets A of $\{1, 2, 3, \ldots, 10\}$ have the property that no two elements of A sum to 11? [1]

10. A polyhedron has faces that are all either triangles or squares. No two square faces share an edge, and no two triangular faces share an edge. What is the ratio of the number of triangular faces to the number of square faces? [1]

11. Find the maximum value of $x + y$, given that $x^2 + y^2 - 3y - 1 = 0$. [1]

12. Let a, b be real numbers such that $\dfrac{ab}{a^2 + b^2} = \dfrac{1}{4}$. Find all possible values of $\dfrac{|a^2 - b^2|}{a^2 + b^2}$. [3]

13. There are 15 stones placed in a line. In how many ways can you mark 5 of these stones so that there are an odd number of stones between any two of the stones you marked? [1]

14. A cafe has 3 tables and 5 individual counter seats. People enter in groups of size between 1 and 4, inclusive, and groups never share a table. A group of more than 1 will always try to sit at a table, but will sit in counter seats if no tables are available. Conversely, a group of 1 will always try to sit at the counter first. One morning, M groups consisting of a total of N people enter and sit down. Then, a single person walks in, and realizes that all the tables and counter seats are occupied by some person or group. What is the minimum possible value of $M + N$? [3]

15. Let ABC be a triangle with circumcenter O such that $AC = 7$. Suppose that the circumcircle of AOC is tangent to BC at C and intersects the line AB at A and F. Let FO intersect BC at E. Compute BE. [3]

16. Let a_1, a_2, a_3, a_4, a_5 be real numbers whose sum is 20. Determine the smallest possible value of

$$\sum_{1 \le i < j \le 5} \lfloor a_i + a_j \rfloor$$ [3]

References

[1] HMMT November 2009, General Test 1. https://www.hmmt.org/www/archive/131

[2] HMMT November 2011, General Test. https://www.hmmt.org/www/archive/151

[3] HMMT February 2013, Team Test. https://www.hmmt.org/www/archive/162