Bijective Combinatorics for Non-Attacking Fillings

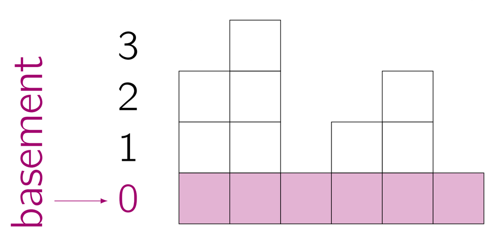
Guilherme Zeus Dantas e Moura (joint work with Olya Mandelshtam)

UNIVERSITY OF WATERLOO

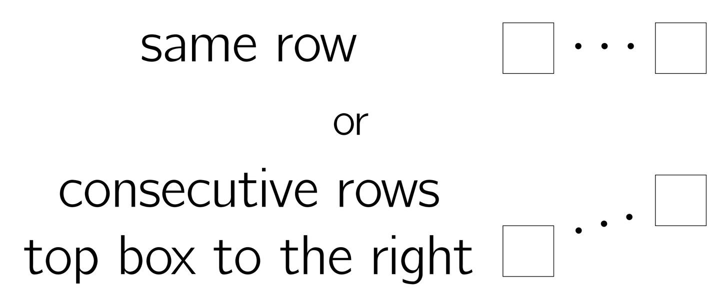
Non-attacking fillings

Augmented skyline diagram of a composition α:

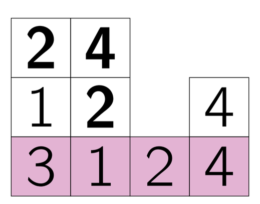
$\alpha = (2\ 3\ 0\ 1\ 2\ 0)$



A pair of boxes is **attacking** if:



A **non-attacking filling** of shape α is a map $T\colon \mathrm{dg}(\alpha) \to [n]$ such that attacking boxes have different entries. The content counts the instances of each i in the filling. The basement of a filling is the permutation $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n] \in \mathfrak{S}_n$ read from the basement.

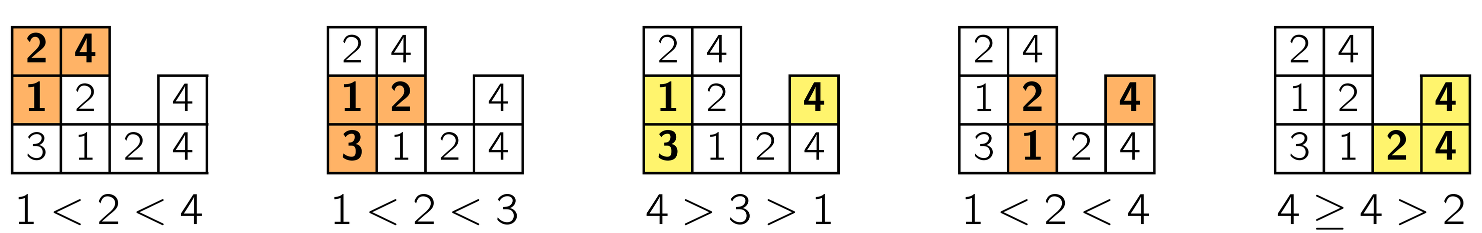


shape $\alpha = (2, 2, 0, 1)$
basement $\sigma = [3, 1, 2, 4]$
content $\beta = (1, 2, 0, 2)$

$x^T = x_1^1 x_2^2 x_3^0 x_4^2$

Major index: $\mathrm{maj}(T) = \sum_{T(u) > T(\text{south of } u)} (1 + \mathrm{leg}(u))$, where $\mathrm{leg}(u) = \#$boxes above u. In the example, $\mathrm{maj}(T) = 1 + 1 + 2 = 4$.

Coinversion number: $\mathrm{coinv}(T) = \#\{$coinversion triples$\}$, where triples are certain L-shaped boxes and the (co)inversion type is the relative order of the entries. In the example, $\mathrm{coinv}(T) = 3$.



Generating function formula for E^σ_α [Fer11]

The permuted basement Macdonald polynomial $E^\sigma_\alpha(x; q, t)$ is

$$E^\sigma_\alpha = \sum_{T \in \mathrm{NAF}(\alpha,\sigma)} x^T q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} \prod_{T(u) \neq T(\text{south of } u)} \frac{1-t}{1 - q^{1+\mathrm{leg}(u)} t^{1+\mathrm{arm}(u)}}$$

From the algebra, permuted basement Macdonald polynomials E^σ_α can also be described as eigenfunctions of modified Cherednik–Dunkl operators, or as transformations of non-symmetric Macdonald polynomials E_α via Demazure–Lusztig operators. Nonsymmetric Macdonald polynomials E_α generalize Demazure characters and atoms, and help understand symmetric Macdonald polynomials P_λ, which extend Jack, Hall–Littlewood, q-Whittaker, and Schur polynomials.

Goal

Overcome some obstacles in using **bijective combinatorics** to study permuted basement Macdonald polynomials E^σ_α:

- constrained objects (non-attacking condition),
- complicated weights,
- different cardinalities.

Change basement via probabilistic bijections

A **probabilistic bijection** [FS24] is a pair of functions $\mathrm{prob}\colon \mathcal{T} \times \mathcal{U} \to A$ and $\mathrm{prob}'\colon \mathcal{U} \times \mathcal{T} \to A$ such that, for all $T \in \mathcal{T}$ and $U \in \mathcal{U}$,

- $\sum_{U \in \mathcal{U}} \mathrm{prob}(T, U) = 1$, $\sum_{T \in \mathcal{T}} \mathrm{prob}'(U, T) = 1$, and
- $\mathrm{wt}(T)\, \mathrm{prob}(T, U) = \mathrm{wt}(U)\, \mathrm{prob}'(U, T)$. *(balance condition)*

This generalizes the notion of weight-preserving bijections. If there exists a probabilistic bijection between $\mathcal{T}$ and $\mathcal{U}$, then

$$\sum_{T \in \mathcal{T}} \mathrm{wt}(T) = \sum_{U \in \mathcal{U}} \mathrm{wt}(U).$$

Theorem 1

Let α be a composition with $\alpha_i = \alpha_{i+1}$ and σ be a permutation. Then,

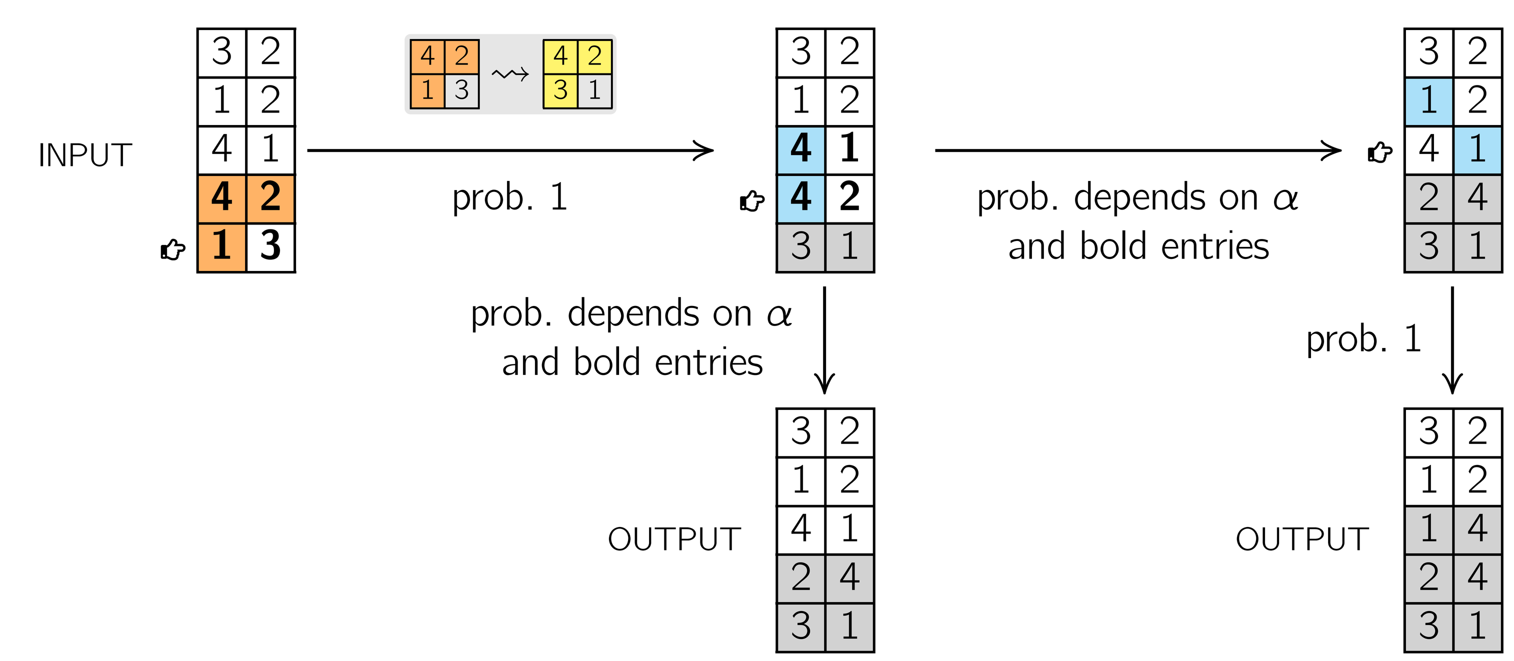
$$E^\sigma_\alpha = E^{\sigma s_i}_\alpha,$$

where $\sigma s_i = [\sigma_1, \ldots, \sigma_{i+1}, \sigma_i, \ldots, \sigma_n]$.

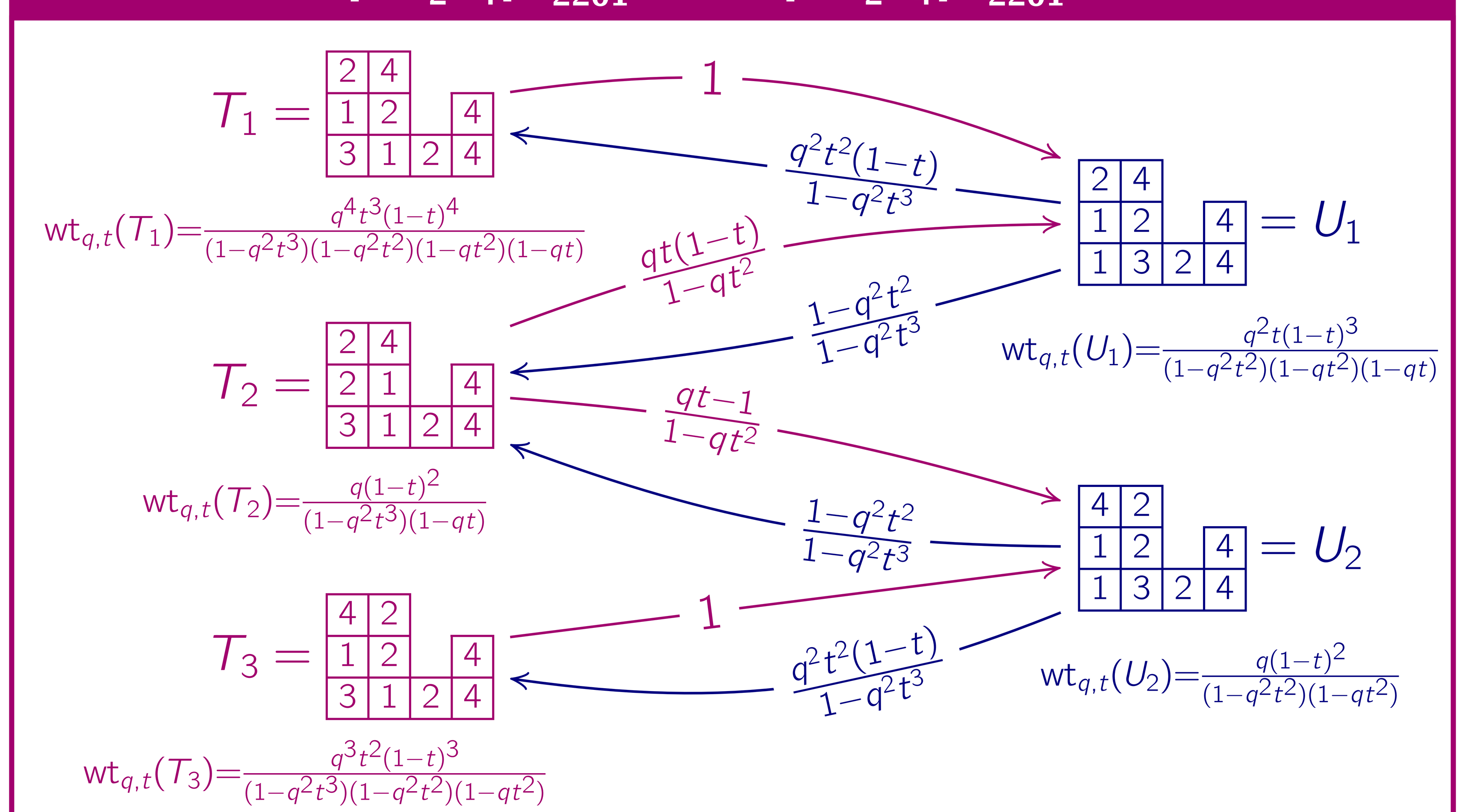
Strategy: Similar to [Man25], construct a probabilistic bijection between

$\mathrm{NAF}(\alpha, \sigma)$ and $\mathrm{NAF}(\alpha, \sigma s_i)$.

The algorithm starts at the bottom row and, depending on the values in each 2×2 square, determines the probabilities for *swapping row entries* and for *moving upward*.



$[x_1 x_2^2 x_4^2] E^{[3,1,2,4]}_{2201}$ vs. $[x_1 x_2^2 x_4^2] E^{[1,3,2,4]}_{2201}$



Change shape via signed fillings

Theorem 2

Let $\alpha \in \mathbb{Z}^n$ with $\alpha_i > \alpha_{i+1}$ and $\sigma \in \mathfrak{S}_n$. Then,

$$E^\sigma_\alpha = E^{\sigma s_i}_{s_i \alpha} + c_{\alpha,\sigma,i}\, E^\sigma_{s_i \alpha},$$

where $c_{\alpha,\sigma,i}(q, t)$ is an explicit rational function in q and t.

Strategy: Adapting [HHL05], use *superization* and a *sign-reversing involution* to rewrite [Fer11]'s formula for E^σ_α as a generating function of **signed fillings** $\mathrm{F}_\pm(\alpha, \sigma)$ (entries from positive and negative integers):

$$E^\sigma_\alpha = d_{\alpha,\sigma} \sum_{T \in \mathrm{F}_\pm(\alpha,\sigma)} x^{|T|} q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} (-t)^{\mathrm{neg}(T)}.$$

Note: a *lot* more objects (since each entry can be positive or negative), but *no* non-attacking condition and *simpler* weights.

Then, adapt [KLO22]'s (maj, inv)-preserving bijection to create bijections

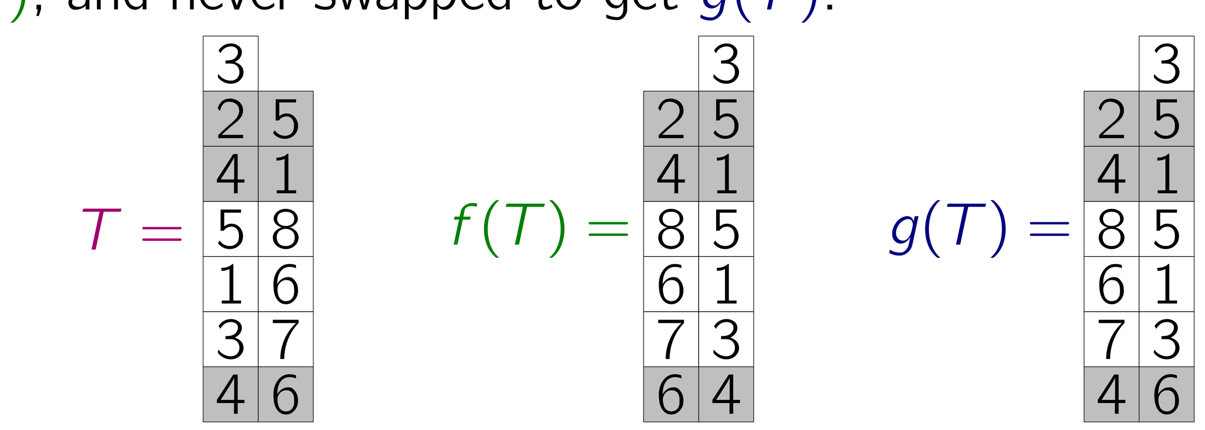
$$\mathrm{F}_\pm(\alpha, \sigma) \xrightarrow{f} \mathrm{F}_\pm(s_i\alpha, \sigma s_i) \quad\quad \mathrm{F}_\pm(\alpha, \sigma) \xrightarrow{g} \mathrm{F}_\pm(s_i\alpha, \sigma)$$

satisfying

$$\mathrm{wt}_\pm(T) = \mathrm{wt}_\pm(f(T)) + c_{\alpha,\sigma,i}\, \mathrm{wt}_\pm(g(T)).$$

Example of bijections f and g

Split rows of T into blocks. Apply a rule to swap or not swap each block. The bottommost block is always swapped to get $f(T)$, and never swapped to get $g(T)$.



References

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