COMBINATORIAL MODELS FOR KEY POLYNOMIALS AND ATOM POLYNOMIALS

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Abstract. We present a survey on some combinatorial models for key polynomials and atom polynomials, in the search for a combinatorial model for the coefficients of the product of two key polynomials in either the key polynomial basis or the atom polynomial basis.

Contents

1. Introduction

The polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$, whose elements are the polynomials with integer coefficients in n variables $x_1, x_2, \ldots, x_n$, is a fundamental algebraic object. Focusing on its additive structure, we can view the polynomial ring as a $\mathbb{Z}$-module; that is, a module over the ring of integers. A basis for a module is a linearly independent set of elements such that every element in the module can be written as an integral linear combination of the basis elements. The canonical basis of the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ is the monomial basis, which consists of monomials $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, indexed by compositions α of length n.

This thesis is concerned with other bases of the polynomial ring: the basis of key polynomials and the basis of atom polynomials. These polynomials

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provide another perspective on the structure of the polynomial ring and have connections to various areas of mathematics.

Before introducing these polynomials, we discuss a well-studied basis of another ring, the ring of symmetric polynomials. The symmetric group $\mathcal{S}_n$ naturally acts on the variables $x_1, x_2, \ldots, x_n$ by permuting them, inducing a corresponding action on the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$. The ring of symmetric polynomials is the subring of $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ consisting of the polynomials that are fixed by the action of $\mathcal{S}_n$. A basis of the ring of symmetric polynomials is the symmetric monomial basis, consisting of the polynomials $m_\lambda = \sum_\alpha x^\alpha$, indexed by partitions λ of length n, where the sum is over all rearrangements α of the parts of λ.

Schur polynomials, indexed by partitions, play a central role in the study of symmetric polynomials, with applications in representation theory, combinatorics, and algebraic geometry. They form a basis for the ring of symmetric polynomials; that is, any symmetric polynomial can be written as an integral linear combination of Schur polynomials. In particular, the product of two Schur polynomials can be written as an integral linear combination of Schur polynomials; that is, given partitions λ and μ,

$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu,$$

where the sum is over partitions ν and the coefficients $c_{\lambda\mu}^\nu$ are integers.

The Littlewood-Richardson rule gives a combinatorial interpretation for the coefficients above, stating that $c_{\lambda\mu}^\nu$ is the number of tableaux with special properties, which is remarkable. In particular, a corollary of the Littlewood-Richardson rule is that the coefficients $c_{\lambda\mu}^\nu$ are nonnegative.

In the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$, the key polynomials play a role analogous to the role of Schur polynomials in the ring of symmetric polynomial. Key polynomials are indexed by compositions of length n, and form a basis of $\mathbb{Z}[x_1, x_2, \ldots, x_n]$. Closely related to the key polynomials are the atom polynomials, also indexed by compositions, which also form a basis of $\mathbb{Z}[x_1, x_2, \ldots, x_n]$.

Since they form a basis, any polynomial can be written as an integral linear combination of the basis elements. In particular, the product of two basis elements can be written as an integral linear combination of the basis elements. The motivation for this thesis is to give a combinatorial interpretation for the coefficients of these products, or more humbly, to discover whether such coefficients are always nonnegative.

The product of two key polynomials does not always have nonnegative coefficients when expanded in the key polynomial basis, for example,

$$\kappa_{01}\kappa_{101} = \kappa_{111} + \kappa_{12} + \kappa_{201} - \kappa_{21}.$$

Similarly, the product of two atom polynomials does not always have nonnegative coefficients when expanded in the atom polynomial basis, for example,

$$A_{01}A_{01} = A_{02} - A_{11}.$$

In contrast, Haglund, Luoto, Mason, and van Willigenburg [HLMW11] introduced a combinatorial model for the coefficients of the product of an atom polynomial and a Schur polynomial in the atom polynomial basis, with the corollary that these coefficients are nonnegative. They also introduced a combinatorial model for the coefficients of the product of a key polynomial and a Schur polynomial in the key polynomial basis, with the corollary that these coefficients are nonnegative.

The open question for this thesis is for a combinatorial model for the coefficients of the product of two key polynomials in either the key polynomial basis or the atom polynomial basis. It is conjectured that the latter coefficients are nonnegative (Conjecture 3.20). This conjecture is the north star of this work. Towards this goal, we introduce multiple combinatorial models for key and atom polynomials, in the hope that the more appropriate model will lead to a solution for the open question.

Section 2 introduces necessary objects, such as permutations, compositions, and partitions, which index the key and atom polynomials. Section 3 introduces multiple families of linear operators on the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$, uses these operators to define key and atom polynomials, and develops on properties and questions related to these polynomials. Section 4 introduces semistandard tableaux, and their right keys, which produce an alternative formula for key and atom polynomials, and also introduces the plactic monoid structure of semistandard tableaux. Section 5 shows that the plactic monoid structure of semistandard tableaux is unable to prove the nonnegativity of the coefficients of the product of two key polynomials. Section 6 introduces Kashiwara crystals, Demazure crystals, and their tensor products, which provide another formula for key and atom polynomials. Finally, Section 7 shows that the crystal structure of the tensor product is unable to prove the nonnegativity of the coefficients of the product of two key polynomials.

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2. Definitions

In this section, we introduce necessary objects, such as permutations, compositions, and partitions. These objects will index the key and atom polynomials, and their properties will be used to define the key and atom polynomials.

2.1. **Permutations.** Let $\mathcal{S}_n$ denote the symmetric group of permutations of $\{1, \dots, n\}$. Let $\mathcal{S}_\infty$ denote the symmetric group of permutations of $\mathbb{Z}_{>0}$ with finitely many non-fixed points. We identify $\mathcal{S}_n$ with the subgroup of $\mathcal{S}_\infty$ that fixes all $i > n$. We write permutations in one-line notation, using brackets. For example, $[4, 1, 3, 2]$ denotes the permutation that maps $1 \mapsto 4$, $2 \mapsto 1$, $3 \mapsto 3$, and $4 \mapsto 2$.

Let σ_i denote the transposition of i and $i+1$, known as the i^{th} **elementary transposition**. Hence,

$$\mathcal{S}_{n+1} = \langle \sigma_1, \sigma_2, \dots, \sigma_n \rangle, \qquad \text{and} \qquad \mathcal{S}_\infty = \langle \sigma_1, \sigma_2, \dots \rangle.$$

For example, $[4, 1, 3, 2] = \sigma_3\sigma_2\sigma_1\sigma_3$. However, the decomposition of a permutation into elementary transpositions is not unique. Indeed, the elementary transpositions satisfy certain relations, stated in Proposition 2.1.

Proposition 2.1 ([Mac91, p. 1])**.** *Let $i, j \in \mathbb{Z}_{>0}$. Then,*

$$\sigma_i^2 = 1, \tag{1a}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{if } |i-j| > 1, \tag{1b}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. \tag{1c}$$

Therefore, it follows that $[4, 1, 3, 2] = \sigma_2\sigma_3\sigma_2\sigma_3\sigma_1\sigma_3$, because

$$\begin{aligned}
\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1\sigma_3 &= \sigma_2\sigma_3\sigma_2\sigma_1\sigma_3\sigma_3 && \text{(by (1b))}\\
&= \sigma_2\sigma_3\sigma_2\sigma_1 && \text{(by (1a))}\\
&= \sigma_3\sigma_2\sigma_3\sigma_1 && \text{(by (1c))}\\
&= \sigma_3\sigma_2\sigma_1\sigma_3 && \text{(by (1b))}\\
&= [4, 1, 3, 2].
\end{aligned}$$

A **word** for a permutation $w \in \mathcal{S}_\infty$ is a sequence $a = a_1a_2 \dots a_k$ of positive integers such that $w = \sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_k}$. The **length** of a word $a = a_1a_2 \dots a_k$ is k, denoted by $\ell(a)$. The **length** of a permutation $w \in \mathcal{S}_\infty$ is the minimum length of a word for w, denoted by $\ell(w)$; that is,

$$\ell(w) = \min\{\ell(a) : a \text{ is a word for } w\}.$$

A word a for w is **reduced** if $\ell(a) = \ell(w)$. Let $\mathsf{RW}(w)$ denote the set of reduced words for w. For example, $\ell([4,1,3,2]) = 4$ and

$$\mathsf{RW}([4,1,3,2]) = \{3213, 3231, 2321\}.$$

Proposition 2.2 (Matsumoto's Theorem, [Mac91, p. 25])**.** *Let $w \in \mathcal{S}_\infty$. Let $\mathsf{GR}(w)$ denote the graph whose vertex set is $\mathsf{RW}(w)$, in which $a \sim b$ is an edge of $\mathsf{GR}(w)$ if either*

- *a is obtained from b by interchanging two consecutive terms i, j such that $|i - j| > 1$ (refer to Equation (1b)), or*
- *a is obtained from b by replacing three consecutive terms i, j, i such that $|i - j| = 1$ by j, i, j (refer to Equation (1c)).*

The graph $\mathsf{GR}(w)$ is connected.

Intuitively, Proposition 2.2 states that, if we know how to write a permutation w as a reduced word, then any other reduced word can be obtained from that one using Equations (1b) and (1c).

Definition 2.3 (Bruhat order on $\mathcal{S}_\infty$, [Mac91, p. 7, inferred from Proposition 1.17])**.** The **Bruhat order** on $\mathcal{S}_\infty$ is the partial order $\leqslant$ such that, for all $w_1, w_2 \in \mathcal{S}_\infty$, we have $w_1 \leqslant w_2$ if a reduced word for w_1 is a subsequence of a reduced word for w_2.

Note that subsequences do not need to appear consecutively. For example, $[2,1,4,3] = \sigma_3\sigma_1 \leqslant \sigma_3\sigma_2\sigma_1\sigma_3 = [4,1,3,2]$.

2.2. **Compositions and Partitions.** A **composition** $\alpha = \alpha_1\alpha_2\ldots$ is an infinite sequence of nonnegative integers with finitely many nonzero terms, indexed by the positive integers. We write $\alpha \models n$ to mean that the sum of the terms of α is n, in which case we say that α is a composition of n. Given two compositions α and β, we write $\alpha \subseteq \beta$ to mean that $\alpha_i \leqslant \beta_i$ for all $i \in \mathbb{Z}_{>0}$. We associate a finite sequence of nonnegative integers with a composition by adding zeros to the end of the sequence.

A **partition** $\lambda = \lambda_1\lambda_2\ldots$ is a weakly decreasing composition. We write $\lambda \vdash n$ to mean that the sum of the terms of λ is n, in which case we say that λ is a partition of n.

For readability, we write compositions and partitions without commas — all examples of compositions and partitions in this thesis have single-digit parts. For example, the sequence $130300\ldots = 1303$ is a composition of 7, but not a partition. The sequence $33100\ldots = 331$ is a partition of 7, hence also a composition of 7.

Let $\mathcal{S}_\infty$ act on the right on the set of compositions by permuting the entries [RS95, p. 109]; that is,

$$\alpha \cdot w = \alpha_{w(1)}\alpha_{w(2)}\cdots.$$

If we view a permutation w as a function $w\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ and a composition α as a function $\alpha\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$, then the action of w on α is given by the functional composition $\alpha \circ w\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$, explaining the choice of a right action. For example,

$$1301 \cdot [4,1,3,2] = 1103,$$

because

$$\begin{aligned}
1 &\xmapsto{[4,1,3,2]} 4 \xmapsto{1301} 1\\
2 &\xmapsto{[4,1,3,2]} 1 \xmapsto{1301} 1\\
3 &\xmapsto{[4,1,3,2]} 3 \xmapsto{1301} 0\\
4 &\xmapsto{[4,1,3,2]} 2 \xmapsto{1301} 3.
\end{aligned}$$

Given a composition α, it is clear that there exists a unique partition, denoted by $\mathsf{sort}\,\alpha$, in the $\mathcal{S}_\infty$-orbit of α, obtained by sorting the entries of α in weakly decreasing order. For example, $\mathsf{sort}(1301) = 311$.

Moreover, given a composition α, there exists a unique permutation $w \in \mathcal{S}_\infty$ of minimal length such that $\alpha = \mathsf{sort}\,\alpha \cdot w$. Such a permutation is called the **shortest sorting permutation** of α. For example, the shortest sorting permutation of 1301 is $[2,1,4,3] = \sigma_1\sigma_3$.

Definition 2.4 (Bruhat order on compositions)**.** The **Bruhat order** on the set of compositions is the partial order $\leqslant$ such that, for all compositions α, β, we have $\alpha \leqslant \beta$ if $\mathsf{sort}\,\alpha = \mathsf{sort}\,\beta$ and the shortest sorting permutation of α is smaller, in the Bruhat order on $\mathcal{S}_\infty$, than the shortest sorting permutation of β.

3. Linear Operators on the Polynomial Ring

In this section, we introduce multiple families of linear operators on the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$. The operators of the first family are induced by the action of the symmetric group $\mathcal{S}_\infty$ on the variables $x_1, x_2, \ldots$. The operators of the second family are the divided difference operators ∂_i, from which the operators of the third family, π_i, and the operators of the fourth family, θ_i, are derived. At the end of this section, key polynomials are defined in terms of the operators π_i, and atom polynomials are defined in terms of the operators θ_i.

The content of this section is based on Macdonald [Mac91, Chapter 2], Reiner and Shimozono [RS95, Section 2], and Pun [Pun16, Chapter 2].

3.1. **Symmetric action on polynomial ring.** The goal of this subsection is to understand the permutations of the symmetric group $\mathcal{S}_\infty$ (or, more particularly, the transpositions) as linear operators on the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$. Given a composition α, we define its **monomial** by

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

For example, $x^{111} = x_1x_2x_3$ and $x^{0002} = x_4^2$.

Recall that $\mathcal{S}_\infty$ is the symmetric group of permutations of $\mathbb{Z}_{>0}$ with finitely many non-fixed points. Let $\mathcal{S}_\infty$ act (on the left) on $\mathbb{Z}[x_1, x_2, \ldots]$ by permuting the variables. Explicitly, if $w \in \mathcal{S}_\infty$ and $f \in \mathbb{Z}[x_1, x_2, \ldots]$, then

$$(wf)(x_1, x_2, \ldots) = f(x_{w(1)}, x_{w(2)}, \ldots).$$

For $i \in \mathbb{Z}_{>0}$, let $\sigma_i \in \mathcal{S}_\infty$ denote the transposition of i and $i+1$. Using the action of $\mathcal{S}_\infty$ on $\mathbb{Z}[x_1, x_2, \ldots]$, we interpret σ_i as a linear operator on $\mathbb{Z}[x_1, x_2, \ldots]$. For example,

$$\sigma_2 \left(x_1x_2 + x_2x_3 + x_3^2 \right) = x_1x_3 + x_2x_3 + x_2^2.$$

Note that σ_i preserves the degree of a monomial.

3.2. **Divided difference operators.** The goal of this subsection is to introduce the divided difference operators ∂_i on the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$, and to understand their properties.

Definition 3.1 (Divided difference operators ∂_i, [Mac91, pp. 23–24])**.** The **divided difference operator** ∂_i on $\mathbb{Z}[x_1, x_2, \ldots]$ is defined by

$$\partial_i f = \frac{f - \sigma_i f}{x_i - x_{i+1}}. \tag{2}$$

We note that, although the right expression of Equation (2) involves polynomial division, the result is always a polynomial; that is, $f - \sigma_i f$ is always a multiple of $x_i - x_{i+1}$. Explicitly, if $f = x_i^r x_{i+1}^s$ we have

$$\partial_i(x_i^r x_{i+1}^s) = \frac{x_i^r x_{i+1}^s - x_{i+1}^r x_i^s}{x_i - x_{i+1}} = \begin{cases} \sum x_i^p y_{i+1}^q & \text{if } r > s, \\ 0 & \text{if } r = s, \\ -\sum x_i^p y_{i+1}^q & \text{if } r < s, \end{cases}$$

where the sum is over (p, q) such that $p + q = r + s - 1$ and $\max(p, q) < \max(r, s)$ [Mac91, pp. 23–24]. For example,

$$\partial_1(x_1^3) = \frac{x_1^3 - \sigma_{1,2}x_1^3}{x_1 - x_2} = \frac{x_1^3 - x_2^3}{x_1 - x_2} = x_1^2 + x_1x_2 + x_2^2.$$

We now present three propositions that describe the behavior of the divided difference operators. Proposition 3.2 describes relations between ∂_i and σ_i, Proposition 3.3 describes relations between ∂_i and ∂_j in an analogous way to Proposition 2.1, and Proposition 3.4 describes how ∂_i acts on a product of two polynomials.

Proposition 3.2 ([Mac91, pp. 23–24])**.** *Let $i \in \mathbb{Z}_{>0}$. Then,*

$$\partial_i \sigma_i = -\partial_i, \tag{3a}$$

$$\sigma_i \partial_i = \partial_i. \tag{3b}$$

Proof. We compute that, for any $f \in \mathbb{Z}[x_1, x_2, \dots]$,

$$\partial_i \sigma_i f = \frac{\sigma_i f - \sigma_i^2 f}{x_i - x_{i+1}} = -\frac{f - \sigma_i f}{x_i - x_{i+1}} = -\partial_i f;$$

and therefore Equation (3a) is true. Equation (3b) is true because, for any $f \in \mathbb{Z}[x_1, x_2, \dots]$, using Equation (1a),

$$\sigma_i \partial_i f = \sigma_i \left(\frac{f - \sigma_i f}{x_i - x_{i+1}} \right) = \frac{\sigma_i f - \sigma_i^2 f}{x_{i+1} - x_i} = \frac{f - \sigma_i f}{x_i - x_{i+1}} = \partial_i f. \qquad \square$$

Note that Equation (3b) means that $\partial_i f$ is symmetric in the variables x_i and x_{i+1}.

Proposition 3.3 ([Mac91, pp. 23–24])**.** *Let $i, j \in \mathbb{Z}_{>0}$. Then,*

$$\partial_i^2 = 0, \tag{4a}$$

$$\partial_i \partial_j = \partial_j \partial_i, \quad \textit{if } |i - j| > 1, \tag{4b}$$

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}. \tag{4c}$$

Proof. Equation (4a) is true because, for any $f \in \mathbb{Z}[x_1, x_2, \dots]$, using Equation (3b),

$$\partial_i^2 f = \frac{\partial_i f - \sigma_i \partial_i f}{x_i - x_{i+1}} = 0.$$

Equation (4b) is true because, for any $f \in \mathbb{Z}[x_1, x_2, \dots]$,

$$\partial_i \partial_j f = \frac{\partial_j f - \sigma_i \partial_j f}{x_i - x_{i+1}} = \frac{f - \sigma_j f - \sigma_i f + \sigma_i \sigma_j f}{(x_i - x_{i+1})(x_j - x_{j+1})},$$

which is invariant under swapping i and j, using Equation (1b). Equation (4c) is true by an analogous but more tedious computation, which we omit here. $\square$

Divided difference operators satisfy a product rule analogous to the Leibniz product rule for derivatives.

Proposition 3.4 (Product rule for divided differences, [Mac91, pp. 23–24])**.** *Given $f, g \in \mathbb{Z}[x_1, x_2, \dots]$, we have*

$$\partial_i(fg) = (\partial_i f) g + (\sigma_i f)(\partial_i g).$$

With the fundamental properties of divided difference operators now established, our next step is to extend these operators from being indexed by integers to being indexed by words, and ultimately to being indexed by permutations.

Recall any permutation can be expressed as a composition of these transpositions in a specific order. Similarly, we extend this concept to divided difference operators. In Definition 3.5, we introduce a notation for the composition of divided difference operators.

Definition 3.5 (Divided difference operators ∂_a, [Mac91, p. 24])**.** For a word $a = a_1 a_2 \dots a_k$, we define

$$\partial_a = \partial_{a_1} \partial_{a_2} \cdots \partial_{a_k}.$$

Recall that the decomposition of a permutation into elementary transpositions is not unique. Even when expressing a permutation as a composition of elementary transpositions with the minimum length, it is still possible that two minimum-length compositions are different, despite representing the same permutation. Proposition 3.6 establishes an analogous statement for divided difference operators. If two words serve as reduced words for the same permutation, their corresponding divided difference operators are identical.

Proposition 3.6 ([Mac91, p. 24])**.** *For a permutation $w \in \mathcal{S}_\infty$, if a and b are two reduced words for w, then $\partial_a = \partial_b$.*

Proof (adapted from [Mac91, p. 25]). From Proposition 2.2, we know that a can be obtained from b by a sequence of operations that are interchanges of two consecutive terms i, j such that $|i - j| > 1$ and replacements of three consecutive terms i, j, i such that $|i-j| = 1$ by j, i, j. Equations (4b) and (4c) imply that those operations preserve the divided difference operator of the word, and therefore $\partial_a = \partial_b$. □

Definition 3.7 (Divided difference operators ∂_w, [Mac91, p. 25])**.** Let $w \in \mathcal{S}_\infty$. Using Proposition 3.6, we can unambiguously define

$$\partial_w = \partial_a,$$

where a is any reduced word for w.

We emphasize the crucial requirement that a must be a reduced word for w, rather than any arbitrary word representing w. In the case where a is a non-reduced word for w, the resulting divided difference operator ∂_a evaluates to the zero operator [Mac91, p. 25]. Although we are drawing a parallel between the divided difference operators and the elementary transpositions, this observation highlights a distinction between them.

3.3. **Isobaric divided difference operators.** We now introduce two kinds of isobaric divided difference operators, denoted by π_i and θ_i, which are other linear operators that preserve the degree of a monomial, differently from the divided difference operators ∂_i. The adjective 'isobaric' is used to emphasize this degree-preserving property. The terminology 'isobaric' is not consistently used in the literature. Some authors use 'isobaric' to refer only to π_i, and some authors do not use the term 'isobaric' at all.

The pace of this subsection is faster than the previous one, because once we define π_i and θ_i, the process of extending them to words and permutations is analogous to the process for ∂_i.

Definition 3.8 (Isobaric divided difference operators π_i and θ_i, [Mac91, p. 27], [Pun16, p. 12])**.** For each $i \in \mathbb{Z}_{>0}$, define the **isobaric divided difference operators** π_i and θ_i on $\mathbb{Z}[x_1, x_2, \dots]$ by

$$\pi_i f = \partial_i(x_i f) \qquad \text{and} \qquad \theta_i f = x_{i+1}(\partial_i f).$$

Proposition 3.9 ([Pun16, p. 14, Proposition 2.1])**.** *Let $i \in \mathbb{Z}_{>0}$. Then,*

$$\pi_i = \mathrm{id} + \theta_i,$$

where id *denotes the identity operator on $\mathbb{Z}[x_1, x_2, \dots]$. Equivalently, for any $f \in \mathbb{Z}[x_1, x_2, \dots]$,*

$$\pi_i f = f + \theta_i f.$$

Proof. Apply the product rule for the divided differences (Proposition 3.4) to the product $x_i f$ and obtain

$$\pi_i f = \partial_i(x_i f) = (\partial_i x_i) f + (\sigma_i x_i)(\partial_i f) = f + x_{i+1}(\partial_i f) = f + \theta_i f,$$

by noting that $\partial_i x_i = 1$ and $\sigma_i x_i = x_{i+1}$. □

Proposition 3.10 ([Mac91, p. 28], [Pun16, pp. 14–16, Propositions 2.1–2.3])**.** *Let $i, j \in \mathbb{Z}_{>0}$. Then,*

$$\pi_i^2 = \pi_i \tag{5a}$$

$$\pi_i \pi_j = \pi_j \pi_i, \quad \text{if } |i-j| > 1 \tag{5b}$$

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}, \tag{5c}$$

and

$$\theta_i^2 = -\theta_i \tag{6a}$$

$$\theta_i \theta_j = \theta_j \theta_i, \quad \text{if } |i-j| > 1 \tag{6b}$$

$$\theta_i \theta_{i+1} \theta_i = \theta_{i+1} \theta_i \theta_{i+1}. \tag{6c}$$

Definition 3.11 (Isobaric divided difference operators π_a and θ_a, [Mac91, p. 28])**.** For a sequence $a = (a_1, a_2, \dots, a_k)$ of positive integers, we define

$$\pi_a = \pi_{a_1} \pi_{a_2} \cdots \pi_{a_k} \qquad \text{and} \qquad \theta_a = \theta_{a_1} \theta_{a_2} \cdots \theta_{a_k}.$$

Proposition 3.12 ([Mac91, p. 28])**.** *For a permutation $w \in \mathcal{S}_\infty$, if*

$$w = \sigma_{a_1} \sigma_{a_2} \cdots \sigma_{a_k} \text{ and } w = \sigma_{b_1} \sigma_{b_2} \cdots \sigma_{b_k}$$

are two reduced words for w, then $\pi_a = \pi_b$ and $\theta_a = \theta_b$.

Proof. The proof is analogous to the proof of Proposition 3.6, and relies on Equations (5b), (5c), (6b), and (6c). □

Definition 3.13 (Isobaric divided difference operators π_w and θ_w, [Mac91, p. 28], [Pun16, p. 22])**.** Let $w \in \mathcal{S}_\infty$. Using Proposition 3.12, we can unambiguously define

$$\pi_w = \pi_a \qquad \text{and} \qquad \theta_w = \theta_a,$$

where a is any reduced word for w.

Recall that Proposition 3.9 describes the relation between π_i and θ_i, indexed by integers. We generalize this relation to π_w and θ_w, indexed by permutations, in Proposition 3.14.

Proposition 3.14 ([Pun16, p. 22, Lemma 2.5])**.** *For any permutation $w \in \mathcal{S}_\infty$, we have*

$$\pi_w = \sum_{w' \leqslant w} \theta_{w'},$$

where the sum is over all permutations $w' \in \mathcal{S}_\infty$ such that $w' \leqslant w$ in the Bruhat order.

This result generalizes Proposition 3.9. By applying Proposition 3.14 to $w = \sigma_i$, we obtain

$$\pi_i = \pi_{\sigma_i} = \sum_{w' \leqslant \sigma_i} \theta_{w'} = \theta_{\mathrm{id}} + \theta_{\sigma_i} = \mathrm{id} + \theta_i,$$

where id represents the identity permutation. Thus, we recover Proposition 3.9 as a special case of Proposition 3.14.

3.4. **Key and Atom Polynomials.** The goal of this subsection is to introduce key and atom polynomials, which are polynomials in the variables $x_1, x_2, \ldots$ that are indexed by compositions. Key polynomials are defined in terms of the isobaric divided difference operators π_w, and atom polynomials are defined in terms of the divided difference operators θ_w. The content of this subsection is based on Reiner and Shimozono [RS95, Section 2] and Mason [Mas09].

Definition 3.15 (Key polynomials, [RS95, p. 109])**.** Let α be a composition. Let $\lambda = \mathsf{sort}\,\alpha$ and w be the shortest permutation such that $\lambda w = \alpha$. The **key polynomial** κ_α, also known as the **Demazure character**, is defined by

$$\kappa_\alpha = \pi_w(x^\lambda).$$

For example,

$$\begin{aligned}
\kappa_{1021} &= \pi_{\sigma_2\sigma_1\sigma_3}(x_1^2x_2x_3) \\
&= \pi_2\pi_1\pi_3(x_1^2x_2x_3) \\
&= \pi_2\pi_1(x_1^2x_2x_3 + x_1^2x_2x_4) \\
&= \pi_2(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2x_4 + x_1x_2^2x_4) \\
&= x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_1^2x_2x_4 + x_1^2x_3x_4 \\
&\quad + x_1x_2^2x_4 + x_1x_2x_3x_4 + x_1x_2^3x_4.
\end{aligned}$$

Definition 3.16 (Atom polynomial)**.** Let α be a composition. Let $\lambda = \mathsf{sort}\,\alpha$ and w be the shortest permutation such that $\lambda w = \alpha$. The **atom polynomial** $\mathcal{A}_\alpha$ is defined by

$$\mathcal{A}_\alpha = \theta_w(x^\lambda).$$

For example,

$$\begin{aligned}
\mathcal{A}_{1021} &= \theta_{\sigma_2\sigma_1\sigma_3}(x_1^2x_2x_3) \\
&= \theta_2\theta_1\theta_3(x_1^2x_2x_3) \\
&= \theta_2\theta_1(x_1^2x_2x_4) \\
&= \theta_2(x_1x_2^2x_4) \\
&= x_1x_2x_3x_4 + x_1x_3^2x_4.
\end{aligned}$$

Key polynomials can be computed by summing atom polynomials. More precisely, Proposition 3.17 gives a formula for a key polynomial as a sum of atom polynomials.

Proposition 3.17 ([LS90], [Pun16, p. 29, Theorem 2.8.1])**.** *Given a composition α,*

$$\kappa_\alpha = \sum_{\beta \leqslant \alpha} \mathcal{A}_\beta,$$

where the order $\leqslant$ is the Bruhat order on compositions defined in Definition 2.4.

Proof. Follows from Proposition 3.14. □

The set of key polynomials and the set of atom polynomials are each a basis for the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$.

Proposition 3.18 ([LS90], [RS95], [Pun16, Theorem 2.8.3])**.** *The sets*

$$\{\kappa_\alpha : \alpha \textit{ is a composition}\} \qquad \textit{and} \qquad \{\mathcal{A}_\alpha : \alpha \textit{ is a composition}\}$$

of keys polynomials and of atom polynomials are each a basis for $\mathbb{Z}[x_1, x_2, \dots]$.

A proof of Proposition 3.18 can be found in Reiner and Shimozono [RS95], and the main idea is to show that

$$\kappa_\alpha = x^\alpha + \sum_{\beta <_{RL} \alpha} c_{\alpha\beta} x^\beta,$$

where $c_{\alpha\beta}$ are nonnegative integers, and $<_{RL}$ is the reverse lexicographic order defined by $\beta <_{RL} \alpha$ if there exists a k satisfying $\beta_i = \alpha_i$ for all $i > k$ and $\beta_k < \alpha_k$. This fact is used to show that the set of key polynomials forms a basis for $\mathbb{Z}[x_1, x_2, \ldots]$. Proposition 3.17 is then used to show that the set of atom polynomials is a basis for $\mathbb{Z}[x_1, x_2, \ldots]$.

There are variations of these results for the case of a finite number of variables.

Proposition 3.19 ([LS90], [Pun16, p. 29, Theorem 2.8.3])**.** *The sets*

$$\{\kappa_\alpha : \alpha \textit{ is a composition of length at most } n\} \textit{ and}$$
$$\{\mathcal{A}_\alpha : \alpha \textit{ is a composition of length at most } n\}$$

are each a basis for $\mathbb{Z}[x_1, x_2, \ldots, x_n]$.

Of particular interest are the polynomials that can be written as a positive linear combination of key polynomials, called **key-positive** polynomials; as well as the polynomials that can be written as a positive linear combination of atom polynomials, called **atom-positive** polynomials. As discussed in Section 1, some key-positivity and atom-positivity results are known.

The product $\kappa_\alpha\kappa_\beta$ of two key polynomials is not always key-positive, for example,

$$\kappa_{01}\kappa_{101} = \kappa_{111} + \kappa_{12} + \kappa_{201} - \kappa_{21};$$

and the product $\mathcal{A}_\alpha\mathcal{A}_\beta$ of two atom polynomials is not always atom-positive, for example,

$$\mathcal{A}_{01}\mathcal{A}_{01} = \mathcal{A}_{02} - \mathcal{A}_{11}.$$

It is an open question whether the product of two key polynomials is always atom-positive, a conjecture that first appeared in an unpublished work of Victor Reiner and Mark Shimozono [Pun16, p. 32].

Conjecture 3.20 ([Pun16, p. 32, Conjecture 1])**.** Let α and β be compositions. Then, $\kappa_\alpha\kappa_\beta$ is atom-positive; that is, there exist nonnegative integers $c^\gamma_{\alpha\beta}$ such that

$$\kappa_\alpha\kappa_\beta = \sum_\gamma c^\gamma_{\alpha\beta}\mathcal{A}_\gamma,$$

where the sum is over all compositions γ.

Pun [Pun16] proved that Conjecture 3.20 is true for α and β of length at most 3; that is, for the product of key polynomials in $\mathbb{Z}[x_1, x_2, x_3]$. Conjecture 3.20 is the north star of this work.

4. Tableaux Combinatorics

In this section, we introduce combinatorial objects called tableaux that can be used for studying key and atom polynomials.

4.1. Diagrams. In the context of diagrams, we think of $\mathbb{Z}^2_{>0}$ as the set of unit boxes in the plane centered at the points with positive integer coordinates. A **diagram** is a subset of $\mathbb{Z}^2_{>0}$. We use matrix-like coordinates, also known as English notation, to graphically represent $\mathbb{Z}^2_{>0}$ as well as diagrams. Figure 1 shows the set $\mathbb{Z}^2_{>0}$, with its elements graphically represented as unit boxes in the plane.

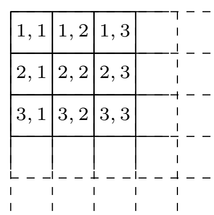


Figure 1. The set $\mathbb{Z}^2_{>0}$, with its elements graphically represented as unit boxes in the plane. The elements of the subset $\{1,2,3\}^2 \subset \mathbb{Z}^2_{>0}$ are labeled with their coordinates.

The set $\mathbb{Z}^2_{>0}$ is partitioned into **rows**, where the i^{th} row is the set $\{i\} \times \mathbb{Z}_{>0}$, and also partitioned into **columns**, where the j^{th} column is the set $\mathbb{Z}_{>0} \times \{j\}$.

The **coordinatewise partial order** on $\mathbb{Z}^2_{>0}$ is the partial order $\leqslant_{\text{coord}}$ given by

$$(i,j) \leqslant_{\text{coord}} (i',j') \qquad \text{if and only if} \qquad i \leqslant i' \quad \text{and} \quad j \leqslant j'.$$

See Figure 2 for a graphical representation of the coordinatewise partial order $\leqslant_{\text{coord}}$ on $\mathbb{Z}^2_{>0}$.

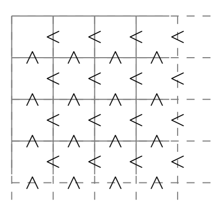


Figure 2. The set $\mathbb{Z}^2_{>0}$, with its elements ordered by the coordinatewise partial order $\leqslant_{\text{coord}}$. Although only the relations between adjacent elements are shown, the relation between an arbitrary pair of elements is determined by the transitivity of the relations in the figure.

The **column order** $\preceq_{\mathrm{col}}$ is the total order given by

$$(i,j) \preceq_{\mathrm{col}} (i',j') \qquad \text{if and only if} \qquad j < j' \quad \text{or} \quad (j = j' \text{ and } i \geqslant i').$$

For example, Figure 3 shows the set $\{1,\ldots,6\}^2 \subset \mathbb{Z}^2_{>0}$ ordered by the column order $\preceq_{\mathrm{col}}$.

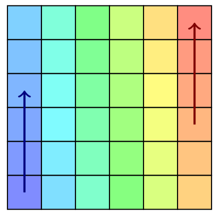


FIGURE 3. The set $\{1,\ldots,6\}^2 \subset \mathbb{Z}^2_{>0}$ ordered by the column order $\preceq_{\mathrm{col}}$. Boxes with cooler colors are smaller than boxes with warmer colors, with respect to the column order $\preceq_{\mathrm{col}}$. The arrows indicate the direction of the orders, pointing from the smaller boxes to the larger boxes.

A **Young diagram** is a diagram $D \subset \mathbb{Z}^2_{>0}$ such that, if $(i,j) \in D$, then $(i',j') \in D$ for all $(i',j') \leqslant_{\mathrm{coord}} (i,j)$. Given a partition λ, **the Young diagram of the partition** λ is the subset of $\mathbb{Z}^2_{>0}$ given by

$$\left\{(i,j) \in \mathbb{Z}^2_{>0} \;:\; j \leqslant \lambda_i\right\}.$$

For example, the Young diagram of 331 is the subset of $\mathbb{Z}^2_{>0}$ given by

$$\Big\{(1,1),(1,2),(1,3), \quad (2,1),(2,2),(2,3), \quad (3,1)\Big\},$$

which is graphically represented in Figure 4.

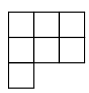


FIGURE 4. The Young diagram of the partition 331.

As an abuse of notation, we use λ to denote both the partition and its Young diagram.

4.2. **Tableaux.** Given a diagram D, a **tableau** of shape D is a map $T\colon D \to \mathbb{Z}_{>0}$. For example, Figure 5 shows multiple examples of tableaux.

Given a tableau T, we define the **weight** of T as the composition $\mathsf{wt}(T) = \alpha = (\alpha_1, \alpha_2, \ldots)$, where α_i is the number of times i appears in T. Moreover, we define the monomial

$$x^T = x^{\mathsf{wt}(T)} = x_1^{\mathsf{wt}(T)_1} x_2^{\mathsf{wt}(T)_2} \cdots .$$

For example, if T is the tableau in Figure 5A then $\mathsf{wt}(T) = 211201$ and $x^T = x_1^2 x_2 x_3 x_4^2 x_6$.

Given a partition λ, a **semistandard tableau** T of shape λ is a tableau $T \colon \lambda \to \mathbb{Z}_{>0}$ such that the entries in each column are strictly increasing, and the entries in each row are weakly increasing. For example, Figure 5B shows a semistandard tableau of shape 331.

4	1	6
3	4	1
2		

(A) A tableau of shape 331 that is not semistandard.

1	1	2
2	3	3
4		

(B) A semistandard tableau of shape 331.

1	1	1
2	2	3

(C) A semistandard tableau of shape 33.

FIGURE 5. Non-semistandard and semistandard tableaux.

4.3. **Plactic Monoid.** In this subsection, we describe an algebraic structure called the plactic monoid, which can be used to study semistandard tableaux. The content of this subsection is based on Fulton [Ful97, Chapter 2].

Recall that a word is a finite sequence of positive integers. Given a tableau T, we define its **column reading word** $\mathsf{w}_{\mathsf{col}}(T)$ as the word obtained by reading the entries of T with respect to the column order $\preceq_{\mathrm{col}}$. For example, the column reading word of the semistandard tableau in Figure 5C is 212131.

Given two words $\mathbf{a}$ and $\mathbf{b}$, we write $\mathbf{ab}$ for the concatenation of $\mathbf{a}$ and $\mathbf{b}$. The concatenation operation is associative and has the empty word $\varnothing$ as its identity element. Hence, the set of words forms a monoid under concatenation.

The **Knuth** or **plactic equivalence** $\sim$ is defined on the set of words by the symmetric, reflexive, and transitive closure of the relations

$$\mathbf{a}xzy\mathbf{b} \sim \mathbf{a}zxy\mathbf{b} \qquad \text{if } x \leqslant y < z, \tag{7a}$$

$$\mathbf{a}yxz\mathbf{b} \sim \mathbf{a}yzx\mathbf{b} \qquad \text{if } x < y \leqslant z. \tag{7b}$$

For example, by taking $\mathbf{a} = 21$, $x = 1$, $y = 2$, $z = 3$, and $\mathbf{b} = 1$, by Equation (7b), we obtain

$$212131 \sim 212311. \tag{8}$$

The plactic equivalence $\sim$ is compatible with the monoid structure of the set of words because, if $\mathbf{a} \sim \mathbf{a}'$ and $\mathbf{b} \sim \mathbf{b}'$, then $\mathbf{ab} \sim \mathbf{ab}' \sim \mathbf{a}'\mathbf{b}'$. This implies that the product operation of plactic equivalence classes can be unambiguously defined by

$$[\mathbf{a}]_\sim \cdot [\mathbf{b}]_\sim = [\mathbf{ab}]_\sim,$$

since we know that the choice of representatives for the plactic equivalence classes does not affect the result of the product. Therefore, the set of plactic equivalence classes of words forms a monoid under the product operation induced by concatenation, which is called the **plactic monoid**.

With this structure in place, Theorem 4.1 relates the plactic monoid to one of the combinatorial objects we are interested in, semistandard tableaux.

Theorem 4.1 (inferred from [Ful97, p. 22])**.** *The map that sends a semistandard tableau T to the plactic equivalence class of its column reading word* $\mathsf{w}_{\mathsf{col}}(T)$

$$T \mapsto [\mathsf{w}_{\mathsf{col}}(T)]_\sim$$

is a bijection between the set of semistandard tableaux and the plactic monoid.

Theorem 4.1 allows us to identify a semistandard tableau with the plactic equivalence class of its column reading word, and vice-versa. Since there is a well-defined product operation on the plactic monoid, we can define a product operation on semistandard tableaux by identifying them with their plactic equivalence classes, as done in Definition 4.2.

Definition 4.2 (inferred from [Ful97, p. 22])**.** Given two tableaux T and U, their product $T \cdot U$ is the unique tableau that is plactic equivalent to the concatenation of the column reading words of T and U; that is,

$$\mathsf{w}_{\mathsf{col}}(T \cdot U) \sim \mathsf{w}_{\mathsf{col}}(T)\mathsf{w}_{\mathsf{col}}(U).$$

The uniqueness of the product in Definition 4.2 follows from Theorem 4.1, which states that there is a unique tableau in the plactic equivalence class of a given word.

Proposition 4.3. *Let T and U be semistandard tableaux. Then,* $\mathsf{wt}(T \cdot U) = \mathsf{wt}(T) + \mathsf{wt}(U)$ *and* $x^{T \cdot U} = x^T x^U$*.*

Proof. The result follows from the observation that two plactic equivalent words have the same weight, since the defining plactic relations in Equations (7a) and (7b) simply permute the entries of the words. $\square$

For example, if

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \text{and} \quad U = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array},$$

whose column reading words are 212 and 311, respectively, then the product of T and U must be the unique tableau whose column reading word is plactic equivalent to 212113. Such tableau is

$$T \cdot U = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \hline\end{array},$$

whose column reading word is 212131, which is plactic equivalent to 212311 by Equation (8). We can moreover check that $x^{T \cdot U} = x_1^3 x_2^2 x_3 = x^T x^U$.

4.4. **Keys.** In this subsection, we introduce keys, which are a particular kind of tableaux that are used to encode compositions in the context of tableaux combinatorics. The content of this subsection is based on Reiner and Shimozono [RS95] and Lascoux and Schützenberger [LS90].

A **key** is a semistandard tableau in which the set of entries in the $(j+1)^{\text{th}}$ column are contained in the set of entries in the j^{th} column, for all $j \in \mathbb{Z}_{>0}$.

1	2	2	6
2	5	6	
4	6		
5			
6			

Figure 6. The key of the composition 130124.

For example, the tableau in Figure 6 is a key, because the set of entries in the fourth column, $\{6\}$, is contained in the set of entries in the third column, $\{2, 6\}$, which is contained in the set of entries in the second column, $\{2, 5, 6\}$, which finally is contained in the set of entries in the first column, $\{1, 2, 4, 5, 6\}$.

Proposition 4.4 ([RS95, p. 111])**.** *The map that sends a key K to the composition $\mathsf{wt}(K)$ is a bijection between the set of keys and the set of compositions. Moreover, the inverse map sends a composition α to the key $\mathsf{key}(\alpha)$, where $\mathsf{key}(\alpha)$ is the semistandard tableau of shape $\mathsf{sort}\,\alpha$ whose first α_j columns contain the entry j, for all $j \in \mathbb{Z}_{>0}$.*

For example, Figure 6 shows the key of the composition 130124, denoted by $\mathsf{key}(130124)$. Note that the shape of $\mathsf{key}(130124)$ is $\mathsf{sort}\,130124 = 43211$.

Proposition 4.5 ([Mas09])**.** *Given two compositions α and β, we have $\alpha \leqslant \beta$ in the Bruhat order if and only if $\mathsf{key}(\alpha) \leqslant \mathsf{key}(\beta)$, by entrywise comparison of tableaux.*

Proposition 4.5 establishes a correspondence between the Bruhat order on compositions and the entrywise order on keys. Since the Bruhat order on compositions is relevant to the study of key and atom polynomials, the entrywise order on keys gives a new perspective on the Bruhat order.

In addition to the aforementioned construction of keys, which serve as a representation of compositions, there exists a way to associate (non-uniquely) a key to each semistandard tableau. Each semistandard tableau T of shape λ is associated to a key, called the **right key** of T, denoted $\mathsf{K}_+(T)$, of shape λ.

To provide an intuitive understanding of the right key of a semistandard tableau, we note that if a semistandard tableau T is already a key, then its right key $\mathsf{K}_+(T)$ is equal to T itself. However, when T is not a key, the right key $\mathsf{K}_+(T)$ is a tableau with entries that are slightly larger than those of T.

The right key of a semistandard tableau was introduced by Lascoux and Schützenberger [LS90]. For an exposition of their definition, we refer the interested reader to Reiner and Shimozono [RS95]. While this thesis does not provide the original definition, we present an algorithm developed by

Willis [Wil13] for constructing the right key of a semistandard tableau. This algorithm, known as the scanning algorithm, is defined and illustrated in Subsection 4.5.

Prior to presenting the algorithm, we elucidate the significance of the right key of a semistandard tableau and its utility in the study of key and atom polynomials.

Theorem 4.6 ([RS95], [LS90]). *The atom polynomial $\mathcal{A}_\alpha$ is*

$$\mathcal{A}_\alpha = \sum_{\substack{\text{semistandard tableaux } T \\ \mathsf{K}_+(T) = \mathsf{key}(\alpha)}} x^T,$$

where the sum is over all semistandard tableaux T of shape $\mathsf{key}(\alpha)$. The key polynomial κ_α is given by

$$\kappa_\alpha = \sum_{\substack{\text{semistandard tableaux } T \\ \mathsf{K}_+(T) \leqslant \mathsf{key}(\alpha)}} x^T,$$

where the sum is over all semistandard tableaux T of shape λ such that $\mathsf{K}_+(T) \leqslant \mathsf{key}(\alpha)$, and $\leqslant$ is the entrywise comparison of tableaux.

4.5. **Scanning Algorithm.** This subsection is based on Willis [Wil13] and provides an algorithm for computing the right key of a semistandard tableau.

Before we present the algorithm, the definition of the earliest weakly increasing subsequence of a word is needed. Given a word $a_1a_2\ldots a_n$, its **earliest weakly increasing subsequence** is the subsequence $a_{i_1}a_{i_2}\ldots a_{i_k}$. where $i_1 = 1$ and the index i_{j+1} is the smallest index such that $i_{j+1} > i_j$ and $a_{i_{j+1}} \geqslant a_{i_j}$.

For example, if $a = 221313243$, then the earliest weakly increasing subsequence of a is $a_1a_2a_4a_6a_8 = 22334$. Moreover, the removal of the earliest weakly increasing subsequence from a yields $a_3a_5a_7a_9 = 1123$.

Algorithm 1 presents the scanning algorithm, described in pseudocode, which features the function RightKey that maps a semistandard tableau T to its corresponding right key $\mathsf{K}_+(T)$.

Example 4.7. We walk through the scanning algorithm to compute

$$\mathsf{K}_+\left(\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline\end{array}\right) = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3&3&3\\\hline\end{array} = \mathsf{key}(303).$$

The algorithm initializes with

$$T = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline\end{array}, \qquad k = 3, \qquad U_0 = \varnothing.$$

The first iteration of the outer loop computes the first column of the output and initializes with

$$j = 1, \qquad m_1 = 2, \qquad T_1 = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline\end{array}, \qquad w_{1,2} = 212131.$$

Algorithm 1 Scanning Algorithm, adapted from Willis [Wil13]

function RightKey(T)
    $k \leftarrow$ number of columns of T
    $U_0 \leftarrow$ empty tableau
    for $j \in \{1, 2, \ldots, k\}$ **do** ▷ outer loop
        $m_j \leftarrow$ number of elements in the j^{th} column of T
        $T_j \leftarrow$ the subtableau of T consisting of columns $j, j+1, \ldots$
        $w_{j,m_j} \leftarrow$ the column reading word of T_j
        for $i \in \{m_j, m_j - 1, \ldots, 1\}$ **do** ▷ inner loop
            $s_{j,i} \leftarrow$ the earliest weakly increasing subsequence of $w_{j,i}$
            $w_{j,i-1} \leftarrow$ removal of $s_{j,i}$ from $w_{j,i}$
        end for
        $c_j \leftarrow$ [last element of $s_{j,1}, \ldots,$ last element of s_{j,m_j}]
        $U_j \leftarrow$ the tableau obtained by adding c_j as a new column to U_{j-1}
    end for
    return U_k
end function

The inner loop recursively computes

$$s_{1,2} = 223, \qquad w_{1,1} = 111,$$
$$s_{1,1} = 111, \qquad w_{1,0} = \varnothing.$$

Then, the algorithm computes $c_1 = (1, 3)$ by taking the last elements of $s_{1,1}$ and $s_{1,2}$, and computes

$$U_1 = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}.$$

The second iteration of the outer loop computes the second column of the output and initializes with

$$j = 2, \qquad m_2 = 2, \qquad T_2 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array}, \qquad w_{2,3} = 2131.$$

The inner loop recursively computes

$$s_{2,2} = 23, \qquad w_{2,1} = 11,$$
$$s_{2,1} = 11, \qquad w_{2,0} = \varnothing.$$

Then, the algorithm computes $c_2 = (1, 3)$ by taking the last elements of $s_{2,1}$ and $s_{2,2}$, and computes

$$U_2 = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}.$$

The third and final iteration of the outer loop computes the third column of the output and initializes with

$$j = 3, \qquad m_3 = 2, \qquad T_3 = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \qquad w_{3,1} = 31.$$

The inner loop recursively computes

$$s_{3,2} = 3, \qquad w_{3,1} = 1,$$
$$s_{3,1} = 1, \qquad w_{3,0} = \varnothing.$$

Then, the algorithm computes $c_3 = (1, 3)$ by taking the last elements of $s_{3,1}$ and $s_{3,2}$, and computes

$$U_3 = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 3 & 3 & 3 \\ \hline \end{array},$$

which the algorithm returns as the output, and hence the claim is proved.

5. Product of Key Polynomials through Keys

In this section, we unsuccessfully attempt to prove Conjecture 3.20 by studying the product of key polynomials through keys. First, we describe the strategy we attempted to use. Then, we present a counterexample that shows the strategy is not viable.

5.1. **Strategy.** Given two compositions α and β, by Corollary 4.6, we have

$$\begin{aligned} \kappa_\alpha \kappa_\beta &= \left(\sum_{\substack{\text{SST } T \\ \mathsf{K}_+(T) \leqslant \mathsf{key}(\alpha)}} x^T \right) \left(\sum_{\substack{\text{SST } U \\ \mathsf{K}_+(U) \leqslant \mathsf{key}(\beta)}} x^U \right) \\ &= \sum_{\substack{\text{SST } T,U \\ \mathsf{K}_+(T) \leqslant \mathsf{key}(\alpha) \\ \mathsf{K}_+(U) \leqslant \mathsf{key}(\beta)}} x^T x^U = \sum_{\substack{\text{SST } T,U \\ \mathsf{K}_+(T) \leqslant \mathsf{key}(\alpha) \\ \mathsf{K}_+(U) \leqslant \mathsf{key}(\beta)}} x^{T \cdot U}. \end{aligned}$$

Therefore, if the multiset

$$\mathsf{P}_{\alpha,\beta} = \left\{ T \cdot U \;:\; \substack{\text{SST } T,U \\ \mathsf{K}_+(T) \leqslant \mathsf{key}(\alpha) \\ \mathsf{K}_+(U) \leqslant \mathsf{key}(\beta)} \right\}$$

can be partitioned into sets of the form

$$\mathsf{A}_\gamma = \left\{ V \;:\; \substack{\text{SST } V \\ \mathsf{K}_+(V) = \mathsf{key}(\gamma)} \right\},$$

then we would be able to conclude that $\kappa_\alpha \kappa_\beta = \sum_\gamma c_{\alpha,\beta}^\gamma A_\gamma$, where $c_{\alpha,\beta}^\gamma$ is the number of times that A_γ appears in such a partition of P.

5.2. **Counterexample.** Consider $\alpha = 021$ and $\beta = 102$, whose corresponding keys are

$$\mathsf{key}(021) = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \text{and} \quad \mathsf{key}(102) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1} \end{array}.$$

Table 1 shows all semistandard tableaux of shape 210, their right keys, and the comparison of their right keys to $\mathsf{key}(021)$ and $\mathsf{key}(102)$.

From Table 1, we observe that there are five tableaux whose right keys are less than or equal to $\mathsf{key}(021)$ and five tableaux whose right keys are less than or equal to $\mathsf{key}(102)$. Therefore, the multiset $\mathsf{P}_{021,102}$ contains twenty-five

Tableau T	$\mathsf{K}_+(T)$	$\mathsf{K}_+(T) \leqslant \mathsf{key}(021)$?	$\mathsf{K}_+(T) \leqslant \mathsf{key}(102)$?
$\ytableaushort{11,2}$	$\ytableaushort{11,2}$	✓	✓
$\ytableaushort{11,3}$	$\ytableaushort{11,3}$	✓	✓
$\ytableaushort{12,2}$	$\ytableaushort{12,2}$	✓	✓
$\ytableaushort{12,3}$	$\ytableaushort{22,3}$	✓	✗
$\ytableaushort{13,2}$	$\ytableaushort{13,3}$	✗	✓
$\ytableaushort{13,3}$	$\ytableaushort{13,3}$	✗	✓
$\ytableaushort{22,3}$	$\ytableaushort{22,3}$	✓	✗
$\ytableaushort{23,3}$	$\ytableaushort{23,3}$	✗	✗

TABLE 1. Semistandard tableaux of shape 210, their right keys, and the comparison of their right keys to $\mathsf{key}(021)$ and $\mathsf{key}(102)$.

tableaux, obtained through all possible products $T \cdot U$ of the tableaux T with $\mathsf{K}_+(T) \leqslant \mathsf{key}(021)$ and U with $\mathsf{K}_+(U) \leqslant \mathsf{key}(102)$. Refer to Definition 4.2 for the product of two tableaux. Table 2 shows all such products.

$T \backslash U$	$\ytableaushort{11,2}$	$\ytableaushort{11,3}$	$\ytableaushort{12,2}$	$\ytableaushort{13,2}$	$\ytableaushort{13,3}$
$\ytableaushort{11,2}$	$\ytableaushort{1111,22}$	$\ytableaushort{1111,23}$	$\ytableaushort{1112,22}$	$\ytableaushort{1113,22}$	$\ytableaushort{1113,23}$
$\ytableaushort{11,3}$	$\ytableaushort{1111,2,3}$	$\ytableaushort{1111,33}$	$\ytableaushort{1112,2,3}$	$\ytableaushort{1113,2,3}$	$\ytableaushort{1113,33}$
$\ytableaushort{12,2}$	$\ytableaushort{111,222}$	$\ytableaushort{111,223}$	$\ytableaushort{1122,22}$	$\ytableaushort{1123,22}$	$\ytableaushort{1133,22}$
$\ytableaushort{12,3}$	$\ytableaushort{111,22,3}$	$\ytableaushort{111,23,3}$	$\ytableaushort{1122,2,3}$	$\ytableaushort{1123,2,3}$	$\ytableaushort{1133,2,3}$
$\ytableaushort{22,3}$	$\ytableaushort{112,22,3}$	$\ytableaushort{113,22,3}$	$\ytableaushort{1222,2,3}$	$\ytableaushort{1223,2,3}$	$\ytableaushort{1233,2,3}$

TABLE 2. The twenty-five products between tableaux with right keys at most $\mathsf{key}(021)$ and tableaux with right keys at most $\mathsf{key}(102)$.

From Table 2 or from Equation (4.3), we observe that one of the tableaux in the multiset $\mathsf{P}_{021,102}$ is

$$\ytableaushort{111,223} = \ytableaushort{12,2} \cdot \ytableaushort{11,3}.$$

From Example 4.7, we know that the right key of this tableau is

$$\mathsf{K}_+\left(\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline\end{array}\right) = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3&3&3\\\hline\end{array} = \mathsf{key}(303).$$

However, the set of all tableaux with key $\mathsf{key}(303)$ is

$$\mathsf{A}_{303} = \left\{\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&3&3\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3&3&3\\\hline\end{array}\right\},$$

which is not a subset of the multiset $\mathsf{P}_{021,102}$, since

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&3&3\\\hline\end{array} \notin \mathsf{P}_{021,102}.$$

Therefore, it is not possible to partition the multiset $\mathsf{P}_{021,102}$ into sets of the form A_γ.

Consequently, the strategy outlined in Subsection 5.1 is not viable for proving Conjecture 3.20, since the strategy fails for the compositions $\alpha = 021$ and $\beta = 102$.

6. Crystal Combinatorics

In this section, we present crystal combinatorics, a combinatorial framework for studying representations of Lie algebras, which allows us to derive key and atom polynomials. However, we focus solely on crystals and establish the connection between crystals and key and atom polynomials directly, without the need for Lie algebras.

The content of this section is based on the book by Bump and Schilling [BS17, Chapters 2, 3, and 13]. We note that Bump and Schilling [BS17] provide a comprehensive introduction to crystals, however, this thesis focuses on the fundamental aspects of crystals relevant to the study of key and atom polynomials, specifically, crystals of type A_{n-1}. For a detailed exposition of crystals of other types, we refer the interested reader to Bump and Schilling [BS17, Chapter 2].

6.1. **Kashiwara Crystals.** In this subsection, we define finite-type Kashiwara crystals of type A_{n-1}, the central algebraic structures in our study of crystal combinatorics.

Let $n \geqslant 2$ be a positive integer. Let $\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{Z}^n$ be the standard basis vectors, and let $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, for $i \in \{1, \dots, n-1\}$.

Definition 6.1 ([BS17, Definition 2.13])**.** A **finite-type Kashiwara crystal** of type A_{n-1} is a set $\mathcal{B}$ with maps

$$\begin{aligned} e_i, f_i &\colon \mathcal{B} \to \mathcal{B} \cup \{0\}, \\ \varepsilon_i, \varphi_i &\colon \mathcal{B} \to \mathbb{Z}, \\ \mathsf{wt} &\colon \mathcal{B} \to \mathbb{Z}^n, \end{aligned}$$

where $i \in \{1, \dots, n-1\}$ and $0 \notin \mathcal{B}$ is an auxiliary element, such that

- for all $u, v \in \mathcal{B}$ and $i \in \{1, \dots, n-1\}$,
$$e_i(u) = v \quad \text{if and only if} \quad f_i(v) = u;$$
- for all $u, v \in \mathcal{B}$ and $i \in \{1, \dots, n-1\}$, if $e_i(u) = v$, then
$$\mathsf{wt}(v) = \mathsf{wt}(u) + \alpha_i, \quad \varepsilon_i(v) = \varepsilon_i(u) - 1, \quad \varphi_i(v) = \varphi_i(u) + 1;$$
- and for all $v \in \mathcal{B}$ and $i \in \{1, \dots, n-1\}$, we have
$$\varphi_i(v) = \mathsf{wt}(v) \cdot \alpha_i + \varepsilon_i(v).$$

The maps e_i and f_i are called the **crystal operators**, the maps ε_i and φ_i are called the **string lengths**, and the map wt is called the **weight map**.

We henceforth refer to finite-type Kashiwara crystals of type A_{n-1} simply as crystals of type A_{n-1}. This is not standard terminology, but we adopt it for brevity.

Definition 6.2. Given two crystals $\mathcal{B}$ and $\mathcal{C}$ of type A_{n-1}, we say that $\mathcal{B}$ is **isomorphic** to $\mathcal{C}$ if there exists a bijection $\beta\colon \mathcal{B} \to \mathcal{C}$ such that, for all $v \in \mathcal{B}$ and $i \in \{1, \dots, n-1\}$,
$$\begin{gathered} \beta(e_i^{\mathcal{B}}(v)) = e_i^{\mathcal{C}}(\beta(v)), \qquad \beta(f_i^{\mathcal{B}}(v)) = f_i^{\mathcal{C}}(\beta(v)), \\ \beta(\varepsilon_i^{\mathcal{B}}(v)) = \varepsilon_i^{\mathcal{C}}(\beta(v)), \qquad \beta(\varphi_i^{\mathcal{B}}(v)) = \varphi_i^{\mathcal{C}}(\beta(v)), \\ \beta(\mathsf{wt}^{\mathcal{B}}(v)) = \mathsf{wt}^{\mathcal{C}}(\beta(v)). \end{gathered}$$

Definition 6.3. Let $\mathcal{B}$ be a crystal of type A_{n-1}. Let $e_i^{\mathcal{B}}, f_i^{\mathcal{B}}, \mathsf{wt}^{\mathcal{B}}, \varepsilon_i^{\mathcal{B}}, \varphi_i^{\mathcal{B}}$ be the crystal operators, weight function, and string lengths of $\mathcal{B}$. Let $\mathcal{C}$ be a subset of $\mathcal{B}$. The **subcrystal of $\mathcal{B}$ induced by $\mathcal{C}$** is the crystal with underlying set $\mathcal{C}$ such that, for all $v \in \mathcal{C}$ and $i \in \{1, \dots, n-1\}$, we have
$$e_i^{\mathcal{C}}(v) = \begin{cases} e_i^{\mathcal{B}}(v) & \text{if } e_i^{\mathcal{B}}(v) \in \mathcal{C}, \\ 0 & \text{otherwise,} \end{cases} \qquad f_i^{\mathcal{C}}(v) = \begin{cases} f_i^{\mathcal{B}}(v) & \text{if } f_i^{\mathcal{B}}(v) \in \mathcal{C}, \\ 0 & \text{otherwise,} \end{cases}$$
$$\varepsilon_i^{\mathcal{C}}(v) = \varepsilon_i^{\mathcal{B}}(v), \qquad \varphi_i^{\mathcal{C}}(v) = \varphi_i^{\mathcal{B}}(v), \qquad \mathsf{wt}^{\mathcal{C}}(v) = \mathsf{wt}^{\mathcal{B}}(v).$$

The **crystal graph** of $\mathcal{B}$ is an edge-labeled directed graph, whose vertices are the elements of $\mathcal{B}$, and in which we draw an edge from u to v labeled i whenever $f_i(u) = v$. See Figure 7 for an example of a crystal graph.

If we ignore the labels and directions of the edges of the crystal graph of $\mathcal{B}$, we obtain a simple undirected graph. Each connected component of this simple graph induces a subcrystal of $\mathcal{B}$, which we call a **full subcrystal** of $\mathcal{B}$.

6.2. **The Crystal $\mathcal{B}_1$ of Type A_{n-1}.** In this subsection, we define the crystal $\mathcal{B}_1$ of type A_{n-1}. This crystal will illustrate the definition in the previous section, and all other crystals in this document will be, in some sense, derived from $\mathcal{B}_1$.

The **crystal $\mathcal{B}_1$ of type A_{n-1}** [BS17, Example 2.19] is the crystal with underlying set

$$\mathcal{B}_1 = \left\{\boxed{1}, \boxed{2}, \boxed{3}, \ldots, \boxed{n}\right\},$$

such that, for all $k \in \{1, \ldots, n\}$ and all $i \in \{1, \ldots, n-1\}$,

$$e_i\left(\boxed{k}\right) = \begin{cases} \boxed{k{-}1} & \text{if } k = i+1, \\ 0 & \text{otherwise,} \end{cases} \qquad f_i\left(\boxed{k}\right) = \begin{cases} \boxed{k{+}1} & \text{if } k = i, \\ 0 & \text{otherwise,} \end{cases}$$

$$\varepsilon_i\left(\boxed{k}\right) = \begin{cases} 1 & \text{if } k = i+1, \\ 0 & \text{otherwise,} \end{cases} \qquad \varphi_i\left(\boxed{k}\right) = \begin{cases} 1 & \text{if } k = i, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathsf{wt}\left(\boxed{k}\right) = \mathbf{e}_k.$$

Figure 7 shows the crystal graph of $\mathcal{B}_1$ of type A_{n-1}.

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \cdots \boxed{n{-}1} \xrightarrow{n-1} \boxed{n}$$

FIGURE 7. The crystal graph of $\mathcal{B}_1$ of type A_{n-1}.

6.3. **Tensor Product of Crystals.** In this subsection, we define the tensor product of crystals, which is a fundamental operation in crystal combinatorics. The crystals that we explore in this thesis are derived from the tensor product of B_1 with itself.

Let $\mathcal{B}$ and $\mathcal{C}$ be crystals of type A_{n-1}. The **tensor product $\mathcal{B} \otimes \mathcal{C}$** [BS17, p. 18] is a crystal of type A_{n-1} with the Cartesian product $\mathcal{B} \times \mathcal{C}$ as the underlying set. The ordered pair (v, u) is denoted by $v \otimes u$. The weight function is given by

$$\mathsf{wt}(v \otimes u) = \mathsf{wt}(v) + \mathsf{wt}(u),$$

for all $v \in \mathcal{B}$ and $u \in \mathcal{C}$. The crystal operators are given by

$$e_i(v \otimes u) = \begin{cases} e_i(v) \otimes u & \text{if } \varphi_i(u) < \varepsilon_i(v), \\ v \otimes e_i(u) & \text{if } \varphi_i(u) \geqslant \varepsilon_i(v), \end{cases}$$

$$f_i(v \otimes u) = \begin{cases} f_i(v) \otimes u & \text{if } \varphi_i(u) \leqslant \varepsilon_i(v), \\ v \otimes f_i(u) & \text{if } \varphi_i(u) > \varepsilon_i(v), \end{cases}$$

for all $v \in \mathcal{B}$ and $u \in \mathcal{C}$, where we let $v \otimes 0 = 0 \otimes u = 0$. The string lengths are given by

$$\varepsilon_i(v \otimes u) = \varepsilon_i(u) + \max\{0, \varepsilon_i(v) - \varphi_i(u)\}$$
$$\varphi_i(v \otimes u) = \varphi_i(v) + \max\{0, \varphi_i(u) - \varepsilon_i(v)\},$$

for all $v \in \mathcal{B}$ and $u \in \mathcal{C}$. Bump and Schilling [BS17, Proposition 2.29] show that $\mathcal{B} \otimes \mathcal{C}$ is indeed a crystal of type A_{n-1}.

Example 6.4. Consider the tensor product $\mathcal{B}_1 \otimes \mathcal{B}_1$ of type A_1. The underlying set is

$$\mathcal{B}_1 \otimes \mathcal{B}_1 = \left\{ \boxed{1} \otimes \boxed{1}, \boxed{1} \otimes \boxed{2}, \boxed{2} \otimes \boxed{1}, \boxed{2} \otimes \boxed{2} \right\},$$

the weight function is given by

$$\begin{aligned}
\mathsf{wt}\left(\boxed{1} \otimes \boxed{1}\right) &= \mathsf{wt}\left(\boxed{1}\right) + \mathsf{wt}\left(\boxed{1}\right) = (1,0) + (1,0) = (2,0), \\
\mathsf{wt}\left(\boxed{1} \otimes \boxed{2}\right) &= \mathsf{wt}\left(\boxed{1}\right) + \mathsf{wt}\left(\boxed{2}\right) = (1,0) + (0,1) = (1,1), \\
\mathsf{wt}\left(\boxed{2} \otimes \boxed{1}\right) &= \mathsf{wt}\left(\boxed{2}\right) + \mathsf{wt}\left(\boxed{1}\right) = (0,1) + (1,0) = (1,1), \\
\mathsf{wt}\left(\boxed{2} \otimes \boxed{2}\right) &= \mathsf{wt}\left(\boxed{2}\right) + \mathsf{wt}\left(\boxed{2}\right) = (0,1) + (0,1) = (0,2),
\end{aligned}$$

the crystal operators are given by

$$\begin{aligned}
e_1\left(\boxed{1} \otimes \boxed{1}\right) &= 0, & f_1\left(\boxed{1} \otimes \boxed{1}\right) &= \boxed{1} \otimes \boxed{2}, \\
e_1\left(\boxed{1} \otimes \boxed{2}\right) &= \boxed{1} \otimes \boxed{1}, & f_1\left(\boxed{1} \otimes \boxed{2}\right) &= \boxed{2} \otimes \boxed{2}, \\
e_1\left(\boxed{2} \otimes \boxed{1}\right) &= 0, & f_1\left(\boxed{2} \otimes \boxed{1}\right) &= 0, \\
e_1\left(\boxed{2} \otimes \boxed{2}\right) &= \boxed{1} \otimes \boxed{2}, & f_1\left(\boxed{2} \otimes \boxed{2}\right) &= 0,
\end{aligned}$$

and the string lengths are given by

$$\begin{aligned}
\varepsilon_1\left(\boxed{1} \otimes \boxed{1}\right) &= 0, & \varphi_1\left(\boxed{1} \otimes \boxed{1}\right) &= 2, \\
\varepsilon_1\left(\boxed{1} \otimes \boxed{2}\right) &= 1, & \varphi_1\left(\boxed{1} \otimes \boxed{2}\right) &= 1, \\
\varepsilon_1\left(\boxed{2} \otimes \boxed{1}\right) &= 0, & \varphi_1\left(\boxed{2} \otimes \boxed{1}\right) &= 0, \\
\varepsilon_1\left(\boxed{2} \otimes \boxed{2}\right) &= 2, & \varphi_1\left(\boxed{2} \otimes \boxed{2}\right) &= 0.
\end{aligned}$$

The crystal graph is shown in Figure 8. We dive deeper into two computations of the crystal operators. For example, we compute $f_1\left(\boxed{1} \otimes \boxed{1}\right)$. Since $\varphi_1(\boxed{1}) = 1 > 0 = \varepsilon_1(\boxed{1})$,

$$f_1\left(\boxed{1} \otimes \boxed{1}\right) = \boxed{1} \otimes f_1(\boxed{1}) = \boxed{1} \otimes \boxed{2}.$$

Similarly, we compute $f_1\left(\boxed{2} \otimes \boxed{1}\right)$. Since $\varphi_1(\boxed{2}) = 0 \leqslant 0 = \varepsilon_1(\boxed{1})$,

$$f_1\left(\boxed{2} \otimes \boxed{1}\right) = f_1(\boxed{2}) \otimes \boxed{1} = 0 \otimes \boxed{1} = 0.$$

$$\begin{tikzcd}
\boxed{1} \otimes \boxed{1} \arrow[d, "1"] & \\
\boxed{1} \otimes \boxed{2} \arrow[d, "1"] & \boxed{2} \otimes \boxed{1} \\
\boxed{2} \otimes \boxed{2} &
\end{tikzcd}$$

FIGURE 8. The crystal graph of $\mathcal{B}_1 \otimes \mathcal{B}_1$ of type A_1.

Proposition 6.5 ([BS17, Proposition 2.32])**.** *Let $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be crystals of type A_{n-1}. Then, $(\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}$ is isomorphic to $\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})$.*

Proposition 6.5 allows us to define, up to isomorphism, the crystal

$$\mathcal{B}_1^{\otimes k} = \underbrace{\mathcal{B}_1 \otimes \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_1}_{k \text{ times}}.$$

The crystal operators of $\mathcal{B}_1^{\otimes k}$ can be computed using the signature rule [BS17, Section 2.4]. The signature rule provides a systematic way to determine which entry i is changed to $i+1$ when the crystal operator f_i is applied, or whether the operator cannot be applied resulting in 0. For a detailed explanation of the signature rule, we refer the interested reader to Bump and Schilling [BS17, Section 2.4].

For example, Figure 9 shows the crystal graph of $\mathcal{B}_1^{\otimes 3}$ of type A_2. Note that $\mathcal{B}_1^{\otimes 3}$ is partitioned into four full subcrystals, one with 10 elements, two with 8 elements, and one with 1 element. Moreover, the two full subcrystals with 8 elements are isomorphic to each other.

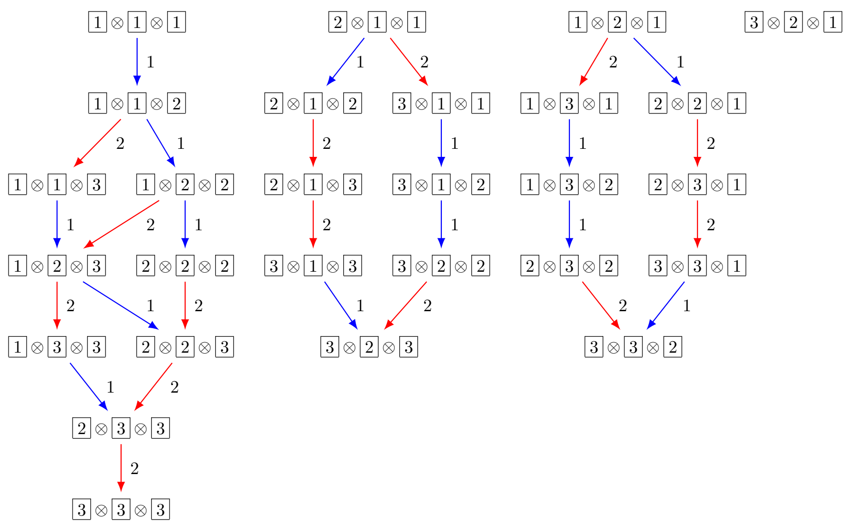


FIGURE 9. The crystal graph of $\mathcal{B}_1^{\otimes 3}$ of type A_2.

6.4. **Crystals of Tableaux.** In this subsection, we define crystals indexed by partitions, whose elements are semistandard tableaux. We have already encountered one of these crystals, $\mathcal{B}_1$, which is indexed by the partition whose only part is 1. In general, the crystal $\mathcal{B}_\lambda$ of type A_{n-1} indexed by a partition λ is defined as the set of semistandard tableaux of shape λ with entries in $\{1, 2, \ldots, n\}$. The crystal structure of $\mathcal{B}_\lambda$ is inherited from the crystal $\mathcal{B}_1^{\otimes|\lambda|}$ of type A_{n-1}, where $|\lambda|$ is the number of boxes in λ.

The content of this subsection is based on Bump and Schilling [BS17, Chapter 3]. We note that the construction of crystals of tableaux is based on row reading words, while our construction will be based on column reading words, for consistency with the rest of the thesis. Bump and Schilling [BS17, Chapter 6] shows that the two constructions are isomorphic.

As a set, $\mathcal{B}_\lambda$ is the set of semistandard tableaux of shape λ with entries in $\{1, 2, \ldots, n\}$. Let $T \in \mathcal{B}_\lambda$. Let $a_1 a_2 \ldots a_k$ be the column reading word of $T \in \mathcal{B}_\lambda$. We associate T to the element

$$\boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_k} \in \mathcal{B}_1^{\otimes|\lambda|}.$$

Bump and Schilling [BS17, Chapter 3] claim that the image of this map forms a full subcrystal of $\mathcal{B}_1^{\otimes|\lambda|}$. The crystal structure of $\mathcal{B}_\lambda$ is given by forcing the map defined above to be a crystal isomorphism between $\mathcal{B}_\lambda$ and this full subcrystal of $\mathcal{B}_1^{\otimes|\lambda|}$.

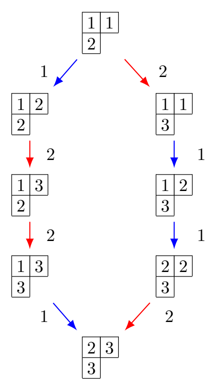


FIGURE 10. The crystal graph of $\mathcal{B}_{21}$ of type A_2.

Example 6.6. We compute the crystal $\mathcal{B}_{21}$ of type A_2. As a set, $\mathcal{B}_{21}$ is the set of semistandard tableaux of shape 21, hence its elements are

$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 & 1 \\ 3 \end{ytableau}, \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau},$$
$$\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}, \quad \begin{ytableau} 1 & 3 \\ 3 \end{ytableau}, \quad \begin{ytableau} 2 & 2 \\ 3 \end{ytableau}, \quad \begin{ytableau} 2 & 3 \\ 3 \end{ytableau}.$$

Under the map from $\mathcal{B}_{21}$ to $\mathcal{B}_1^{\otimes 3}$, these elements are mapped, respectively, to

$$\boxed{2}\otimes\boxed{1}\otimes\boxed{1}, \quad \boxed{3}\otimes\boxed{1}\otimes\boxed{1}, \quad \boxed{2}\otimes\boxed{1}\otimes\boxed{2}, \quad \boxed{3}\otimes\boxed{1}\otimes\boxed{2},$$
$$\boxed{2}\otimes\boxed{1}\otimes\boxed{3}, \quad \boxed{3}\otimes\boxed{1}\otimes\boxed{3}, \quad \boxed{3}\otimes\boxed{2}\otimes\boxed{2}, \quad \boxed{3}\otimes\boxed{2}\otimes\boxed{3}.$$

From Figure 9, we observe that these eight elements form a full subcrystal of $\mathcal{B}_1^{\otimes 3}$. The crystal structure of $\mathcal{B}_{21}$ is given by forcing the map above to be a crystal isomorphism between $\mathcal{B}_{21}$ and this subcrystal. Figure 10 shows the crystal graph of $\mathcal{B}_{21}$ of type A_2.

6.5. **Demazure Crystals.** Given a crystal $\mathcal{B}$ of type A_{n-1}, define the operator $\mathcal{D}_i$ on a subset $X \subseteq \mathcal{B}$ by

$$\mathcal{D}_i X = \{x \in \mathcal{B} : x = f_i^k(y) \text{ for some } y \in X \text{ and } k \in \mathbb{Z}_{\geqslant 0}\},$$

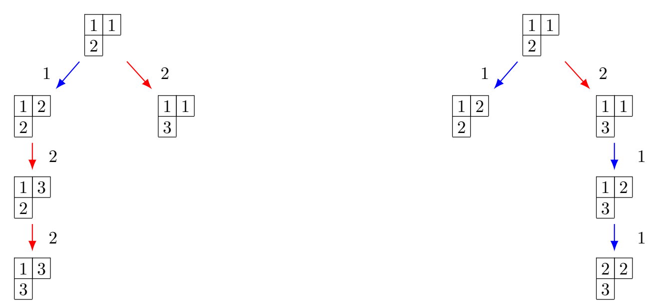
where f_i^k denotes the k-fold application of the operator f_i.

Let λ be a partition with at most n parts. Let $w \in \mathcal{S}_n$ be a permutation, and let $a_1 a_2 \dots a_\ell$ be a reduced word for w. The **Demazure crystal** $\mathcal{B}_\lambda(w)$ of type A_{n-1} is the subcrystal of $\mathcal{B}_\lambda$ of type A_{n-1} induced by

$$\mathcal{D}_{a_1}\mathcal{D}_{a_2}\cdots\mathcal{D}_{a_\ell}\{T_\lambda\},$$

where T_λ is the tableau of shape λ such that the entries in the i^{th} row are i.

Theorem 6.7 ([BS17, inferred from Theorem 13.5])**.** *The definition of $\mathcal{B}_\lambda(w)$ is independent of the choice of reduced word for w.*



(A) The crystal graph of $\mathcal{B}_{21}(\sigma_2\sigma_1)$. (B) The crystal graph of $\mathcal{B}_{21}(\sigma_2\sigma_1)$.

Figure 11. Crystal graphs of two Demazure subcrystals of $\mathcal{B}_{21}$ of type A_2.

Example 6.8. We compute the Demazure crystal $\mathcal{B}_{21}(\sigma_2\sigma_1)$ of type A_2. Refer to Figure 10 for the crystal graph of $\mathcal{B}_{21}$ of type A_2, which we know that $\mathcal{B}_{21}(\sigma_2\sigma_1)$ is a subcrystal of. Note that $\sigma_2\sigma_1$ has reduced word 21. Therefore, $\mathcal{B}_{21}(\sigma_2\sigma_1)$ is the subcrystal of $\mathcal{B}_{21}$ induced by

$$\mathcal{D}_2\mathcal{D}_1\left\{\ytableaushort{11,2}\right\} = \mathcal{D}_2\left\{\ytableaushort{11,2}, \ytableaushort{12,2}\right\} = \left\{\ytableaushort{11,2}, \ytableaushort{11,3}, \ytableaushort{12,2}, \ytableaushort{13,2}, \ytableaushort{13,3}\right\}.$$

In simple terms, starting with the tableau $\ytableaushort{11,2}$, we apply the operator f_1 any number of times (including zero), and then apply the operator f_2 any number of times (including zero). Refer to Figure 11A for the crystal graph of $\mathcal{B}_{21}(\sigma_2\sigma_1)$.

Example 6.9. We compute the Demazure crystal $\mathcal{B}_{21}(\sigma_1\sigma_2)$ of type A_2. Note that $\sigma_1\sigma_2$ has reduced word 12. Therefore, $\mathcal{B}_{21}(\sigma_1\sigma_2)$ is the subcrystal of $\mathcal{B}_{21}$

induced by

$$\mathcal{D}_1\mathcal{D}_2\left\{\ytableaushort{11,2}\right\} = \mathcal{D}_1\left\{\ytableaushort{11,2}, \ytableaushort{11,3}\right\} = \left\{\ytableaushort{11,2}, \ytableaushort{12,2}, \ytableaushort{11,3}, \ytableaushort{12,3}, \ytableaushort{13,3}\right\}.$$

In simple terms, starting with the tableau $\ytableaushort{11,2}$, we apply the operator f_2 any number of times (including zero), and then apply the operator f_1 any number of times (including zero). Refer to Figure 11B for the crystal graph of $\mathcal{B}_{21}(\sigma_1\sigma_2)$.

Let λ be a partition with at most n parts. Let $w \in \mathcal{S}_n$ be a permutation. The **Demazure crystal atom** $\mathcal{A}_\lambda(w)$ of type A_{n-1} [Arm23, Definition 3.0.1] is the subcrystal of $\mathcal{B}_\lambda$ of type A_{n-1} induced by

$$\mathcal{B}_\lambda(w) \ \setminus \ \bigcup_{v<w} \mathcal{B}_\lambda(v),$$

where $<$ denotes the Bruhat order on $\mathcal{S}_n$.

For example, Figure 12 shows the crystal graph of $\mathcal{A}_{21}(\sigma_2\sigma_1)$ of type A_2.

$$\begin{array}{c} \ytableaushort{13,2} \\ \big\downarrow{\scriptstyle 2} \\ \ytableaushort{13,3} \end{array}$$

Figure 12. The crystal graph of $\mathcal{A}_{21}(\sigma_2\sigma_1)$.

Proposition 6.10 ([Arm23, Theorem 3.0.2])**.** *Let λ be a partition with at most n parts, and let $w \in \mathcal{S}_n$ be a permutation. Then,*

$$\mathcal{B}_\lambda(w) = \bigsqcup_{\substack{w' \in \mathcal{S}_n^\lambda \\ w' \leqslant w}} \mathcal{A}_\lambda(v),$$

where $\mathcal{S}_n^\lambda$ is the set of permutations w that are the shortest sorting permutations of λw.

6.6. **Demazure Characters and Demazure Atoms.** Given a crystal $\mathcal{B}$ of type A_{n-1}, we define its **character** as

$$\chi(\mathcal{B}) = \sum_{v \in \mathcal{B}} x^{\mathsf{wt}(v)} \in \mathbb{Z}[x_1, x_2, \ldots, x_n].$$

Finally, we return to the key and atom polynomials. Theorem 6.11 states that the character of a Demazure crystal is a key polynomial, and the character of a Demazure atom is an atom polynomial.

Theorem 6.11 ([BS17, Theorem 13.7], [Arm23, Theorem 3.0.1])**.** *Let λ be a partition with at most n parts, and let $w \in \mathcal{S}_n$ be a permutation. The character of the Demazure crystal $\mathcal{B}_\lambda(w)$ is the key polynomial $\kappa_{\lambda w}$; that is,*

$$\kappa_{\lambda w} = \chi(\mathcal{B}_\lambda(w)).$$

The character of the Demazure atom $\mathcal{A}_\lambda(w)$ is the atom polynomial $A_{\lambda w}$; that is,

$$A_{\lambda w} = \chi(\mathcal{A}_\lambda(w)).$$

Since we are interested in the product of two key polynomials, we are interested in the product of characters of Demazure crystals. Proposition 6.12 allows us to write a product of characters of crystals as the character of the tensor product of those crystals.

Proposition 6.12 ([BS17, Exercise 2.11])**.** *Let $\mathcal{B}, \mathcal{C}$ be crystals of type A_{n-1}. Then*

$$\chi(\mathcal{B} \otimes \mathcal{C}) = \chi(\mathcal{B})\chi(\mathcal{C}).$$

Proof. Recall that, as a set, $\mathcal{B} \otimes \mathcal{C} = \mathcal{B} \times \mathcal{C}$. Moreover, the weight function of $\mathcal{B} \otimes \mathcal{C}$ is given by $\mathsf{wt}(v \otimes u) = \mathsf{wt}(v) + \mathsf{wt}(u)$. Hence,

$$\begin{aligned}\chi(\mathcal{B} \otimes \mathcal{C}) &= \sum_{v\otimes u \in \mathcal{B}\otimes\mathcal{C}} x^{\mathsf{wt}(v\otimes u)} \\ &= \sum_{(v,u)\in\mathcal{B}\times\mathcal{C}} x^{\mathsf{wt}(v)+\mathsf{wt}(u)} = \left(\sum_{v\in\mathcal{B}} x^{\mathsf{wt}(v)}\right)\left(\sum_{u\in\mathcal{C}} x^{\mathsf{wt}(u)}\right) = \chi(\mathcal{B})\chi(\mathcal{C}). \square\end{aligned}$$

7. Tensor Product of Demazure Crystals

In this section, we provide another unsuccessful attempt to prove Conjecture 3.20, by studying the tensor product of Demazure crystals. We describe the strategy we attempted to use, present an example where the strategy works, and finally provide a counterexample where the strategy is not viable.

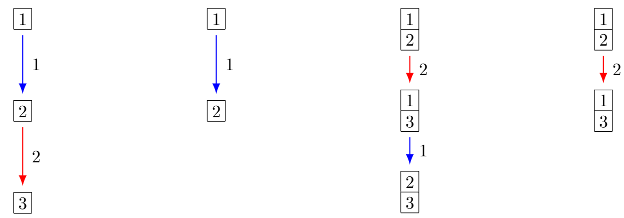
7.1. **Strategy.** Recall that Conjecture 3.20 states that the product of two Demazure key polynomials is a nonnegative linear combination of Demazure atom polynomials. Note that the product $\kappa_{\lambda_1 w_1}\kappa_{\lambda_2 w_2}$ is the character of the tensor product of Demazure crystals $\mathcal{B}_{\lambda_1}(w_1) \otimes \mathcal{B}_{\lambda_2}(w_2)$.

Therefore, if we are able to decompose a tensor product of the form $\mathcal{B}_{\lambda_1}(w_1)\otimes\mathcal{B}_{\lambda_2}(w_2)$ into pieces that are isomorphic to Demazure atom crystals then we would conclude that

$$\kappa_{\lambda_1 w_1}\kappa_{\lambda_2 w_2} = \sum_{\mu,v} c^{\mu v}_{\lambda_1 w_1, \lambda_2 w_2} A_{\mu v}$$

where the coefficient $c^{\mu v}_{\lambda_1 w_1,\lambda_2 w_2}$ is the number of pieces isomorphic to the Demazure atom crystal $\mathcal{A}_\mu(v)$ in the such a decomposition of the tensor product of Demazure crystals $\mathcal{B}_{\lambda_1}(w_1) \otimes \mathcal{B}_{\lambda_2}(w_2)$.

7.2. **Example.** Consider the tensor product of two Demazure crystals $\mathcal{B}_1(\sigma_1)$ and $\mathcal{B}_{11}(\sigma_2)$, of type A_2. Figure 13 shows the crystal graphs of the crystal B_1, its Demazure crystal $\mathcal{B}_1(\sigma_1)$, the crystal $\mathcal{B}_{11}$, and its Demazure crystal $\mathcal{B}_{11}(\sigma_2)$. Figure 14 shows the tensor product $\mathcal{B}_1(\sigma_1) \otimes \mathcal{B}_{11}(\sigma_2)$.



(A) $\mathcal{B}_1$. (B) $\mathcal{B}_1(\sigma_1)$. (C) $\mathcal{B}_{11}$. (D) $\mathcal{B}_{11}(\sigma_2)$.

FIGURE 13. Crystal graphs of two crystals of tableaux and Demazure subcrystals.

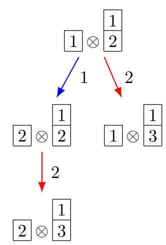


FIGURE 14. The tensor product $\mathcal{B}_1(\sigma_1) \otimes \mathcal{B}_{11}(\sigma_2)$.

We can decompose the tensor product $\mathcal{B}_1 \otimes \mathcal{B}_{11}$ into four pieces, each containing a single element. Ordering the pieces as they appear from top to bottom, then left to right, in Figure 14, these pieces are isomorphic to the Demazure atom crystals $\mathcal{A}_{21}(\mathrm{id})$, $\mathcal{A}_{21}(\sigma_1)$, $\mathcal{A}_{21}(\sigma_2)$, and $\mathcal{A}_{111}(\mathrm{id})$, respectively. The four listed Demazure atom crystals are shown in Figure 15. To check that these pairs of crystals are isomorphic, since each crystal has only one element, it suffices to check that the weight of the only element in a piece is the same as the weight of the only element in the respective Demazure atom crystal.

(A) $\mathcal{A}_{111}(\mathrm{id})$. (B) $\mathcal{A}_{21}(\sigma_2)$. (C) $\mathcal{A}_{21}(\sigma_1)$. (D) $\mathcal{A}_{21}(\mathrm{id})$.

FIGURE 15. Demazure atom crystals of type A_2. Gray nodes are not elements of the subcrystals, but are included for clarity.

As a corollary of this decomposition, we obtain

$$\kappa_{1\sigma_1}\kappa_{11\sigma_2} = A_{111\mathrm{id}} + A_{21\sigma_2} + A_{21\sigma_1} + A_{21\mathrm{id}},$$

or equivalently, in terms of compositions,

$$\kappa_{01}\kappa_{101} = A_{111} + A_{201} + A_{12} + A_{21}.$$

7.3. **Counterexample.** Consider the product of the key polynomials $\kappa_{21\sigma_2\sigma_1}$ and $\kappa_{21\sigma_1\sigma_2}$, of type A_2. The Demazure crystals $\mathcal{B}_{21}(\sigma_2\sigma_1)$ and $\mathcal{B}_{21}(\sigma_1\sigma_2)$ are shown in Figure 11. Their tensor product $\mathcal{B}_{21}(\sigma_2\sigma_1) \otimes \mathcal{B}_{21}(\sigma_1\sigma_2)$ is shown in Figure 16.

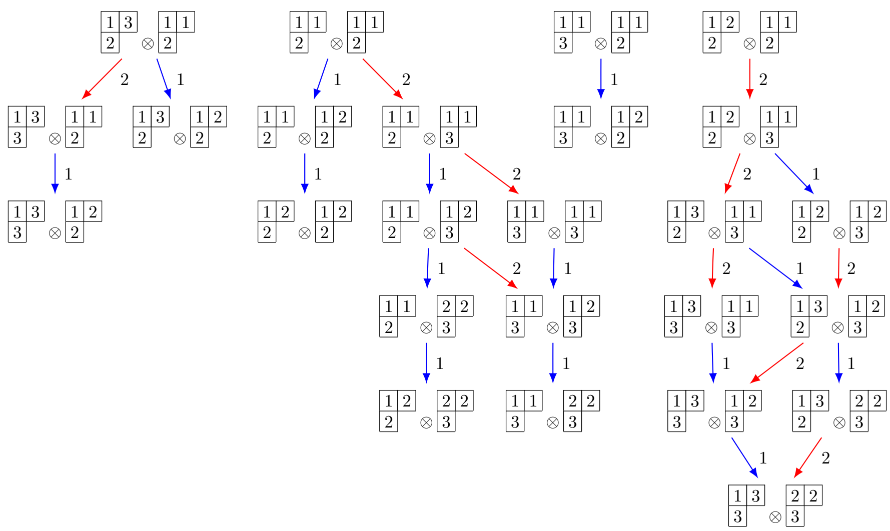


FIGURE 16. The tensor product $\mathcal{B}_{21}(\sigma_2\sigma_1) \otimes \mathcal{B}_{21}(\sigma_1\sigma_2)$.

Using SageMath [Sage; SC], we compute the product of the key polynomials $\kappa_{21\sigma_2\sigma_1}$ and $\kappa_{21\sigma_1\sigma_2}$ to be

$$\begin{aligned}&2A_{321} + 2A_{321\sigma_1} + 2A_{321\sigma_2} + A_{321\sigma_1\sigma_2} + A_{321\sigma_2\sigma_1} + A_{321\sigma_1\sigma_2\sigma_1}\\ &+A_{411} + A_{411\sigma_1} + A_{222} + A_{33} + A_{33\sigma_2} + A_{42} + A_{42\sigma_1} + A_{42\sigma_2}.\end{aligned}$$

Therefore, if there is a decomposition of the tensor product of Demazure crystals $\mathcal{B}_{21}(\sigma_2\sigma_1) \otimes \mathcal{B}_{21}(\sigma_1\sigma_2)$ into pieces that are isomorphic to Demazure crystal atoms, then the decomposition must consist of pieces isomorphic to

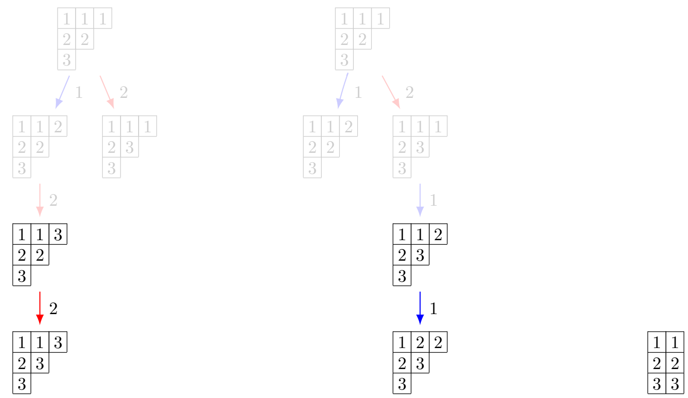
$$\begin{aligned}&2\mathcal{A}_{321}(\mathrm{id}), 2\mathcal{A}_{321}(\sigma_1), 2\mathcal{A}_{321}(\sigma_2), \mathcal{A}_{321}(\sigma_1\sigma_2), \mathcal{A}_{321}(\sigma_2\sigma_1), \mathcal{A}_{321}(\sigma_1\sigma_2\sigma_1),\\ &\mathcal{A}_{411}(\mathrm{id}), \mathcal{A}_{411}(\sigma_1), \mathcal{A}_{222}(\mathrm{id}), \mathcal{A}_{33}(\mathrm{id}), \mathcal{A}_{33}(\sigma_2), \mathcal{A}_{42}(\mathrm{id}), \mathcal{A}_{42}(\sigma_1), \mathcal{A}_{42}(\sigma_2),\end{aligned}$$

where the number in front of each atom indicates the number of times it appears in the decomposition.

To see that such a decomposition is impossible, let's pay attention to the three vertices with weight 222, namely

$$\begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}.$$

These vertices must appear, in some order, in the parts isomorphic to the Demazure crystal atoms $\mathcal{A}_{321}(\sigma_1\sigma_2)$, $\mathcal{A}_{321}(\sigma_2\sigma_1)$, and $\mathcal{A}_{222}(\mathrm{id})$, since these are the three Demazure crystal atoms in the list above that contain a vertex with weight 222. Refer to Figure 17 for the crystal graph of these three Demazure crystal atoms.



(A) $\mathcal{A}_{21}(\sigma_2\sigma_1)$. (B) $\mathcal{A}_{21}(\sigma_1\sigma_2)$. (C) $\mathcal{A}_{222}(\mathrm{id})$.

FIGURE 17. Demazure crystal atoms containing vertices with weight 222. Gray elements are not part of crystal.

On one hand, in two of these three Demazure crystal atoms, the vertex with weight 222 is connected to a vertex of smaller weight. On the other hand, in the tensor product $\mathcal{B}_{21}(\sigma_2\sigma_1) \otimes \mathcal{B}_{21}(\sigma_1\sigma_2)$, two of the three vertices with weight 222 are not connected to any vertex of smaller weight. Therefore, the tensor product $\mathcal{B}_{21}(\sigma_2\sigma_1) \otimes \mathcal{B}_{21}(\sigma_1\sigma_2)$ cannot be decomposed into pieces that are isomorphic to Demazure crystal atoms.

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