S-Legal Index Difference Sequences

Guilherme Zeus Dantas e Moura (Haverford College)

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Joint work with Drew Keisling, Astrid Lilly, Annika Mauro, Steven J. Miller, Matthew Phang, Santiago Velazquez Iannuzzelli

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Zeckendorf's theorem

Let $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Theorem (Zeckendorf)

Every positive integer has a unique decomposition as a sum of non-consecutive Fibonacci numbers.

Zeckendorf decompositions

$$1 = F_1$$
$$2 = F_2$$
$$3 = F_3$$
$$4 = F_3 + F_1$$
$$5 = F_4$$
$$6 = F_4 + F_1$$
$$7 = F_4 + F_2$$
$$8 = F_5$$
$$9 = F_5 + F_1$$
$$10 = F_5 + F_2$$
$$11 = F_5 + F_3$$
$$12 = F_5 + F_3 + F_1$$
$$13 = F_6$$
$$14 = F_6 + F_1$$
$$15 = F_6 + F_2$$
$$16 = F_6 + F_3$$
$$17 = F_6 + F_3 + F_1$$
$$18 = F_6 + F_4$$
$$19 = F_6 + F_4 + F_1$$
$$20 = F_6 + F_4 + F_2$$
$$21 = F_7$$
$$22 = F_7 + F_1$$
$$23 = F_7 + F_2$$
$$24 = F_7 + F_3$$
$$25 = F_7 + F_3 + F_1$$
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$$31 = F_7 + F_5 + F_2$$
$$32 = F_7 + F_5 + F_3$$
$$33 = F_7 + F_5 + F_3 + F_1$$
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$$35 = F_8 + F_1$$
$$36 = F_8 + F_2$$
$$37 = F_8 + F_3$$
$$38 = F_8 + F_3 + F_1$$
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$$40 = F_8 + F_4 + F_1$$
$$41 = F_8 + F_4 + F_2$$
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$$44 = F_8 + F_5 + F_2$$
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Zeckendorf decompositions

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Redefining Fibonacci sequence

Defining Property (Fibonacci sequence)

F_n is the smallest positive integer which **cannot** be written as a sum of non-consecutive terms in $\{F_1, \ldots, F_{n-1}\}$.

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2^n is the smallest positive integer which **cannot** be written as a sum of terms in $\{2^0, 2^1, \ldots, 2^{n-1}\}$.

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Defining Property (Powers of 2)

2^n is the smallest positive integer which **cannot** be written as a sum of terms in $\{2^0, 2^1, \ldots, 2^{n-1}\}$.

Template

a_n is the smallest positive integer which **cannot** be written as an "allowed" sum of terms in $\{a_1, a_2, \ldots, a_{n-1}\}$.

Interpretation of consecutiveness

Fibonacci sequence: "allowed" means "non-consecutive."

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We put the Fibonacci sequence in a 1D array of boxes.

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Fibonacci sequence: “allowed” means “non-adjacent.”

Why stop at 1D?

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But, an 1D array of boxes is boring...

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But, an 1D array of boxes is boring...

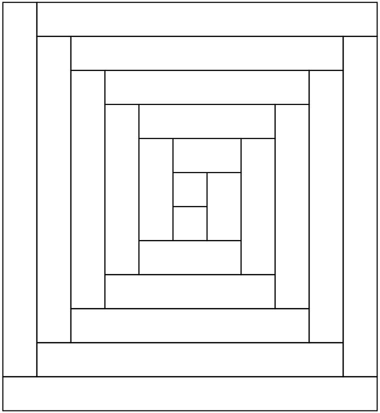
Catral, Ford, Harris, Miller, and Nelson decided to use the Fibonacci spiral instead, and defined a new sequence.

The Fibonacci... spiral?

Me: Mom, can we get a Fibonacci spiral?

Mom: No, we have a Fibonacci spiral at home.

At home:



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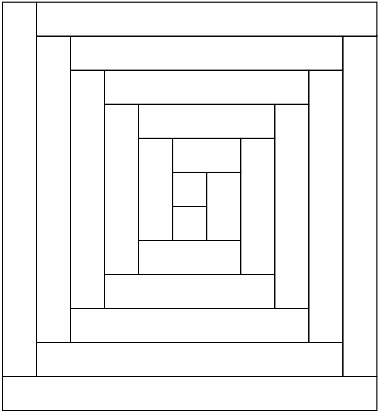
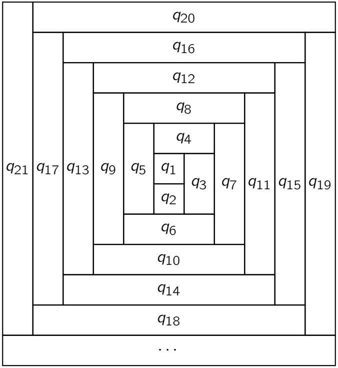


The Fibonacci quilt. For the purposes of adjacency, same as the Fibonacci spiral.

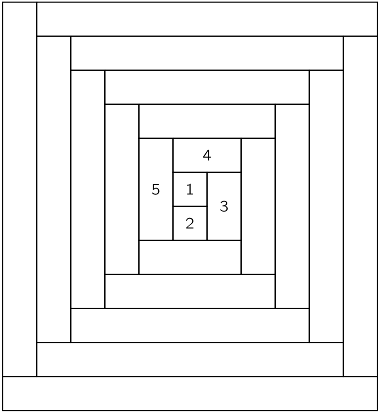
The Fibonacci quilt sequence

Definition (Fibonacci quilt sequence; CFHMN, 2020)

Let q_n be the smallest positive integer which **cannot** be written as as a sum of non-adjacent terms in $\{q_1, \ldots, q_{n-1}\}$. Two terms are adjacent if their boxes share an edge in the quilt.



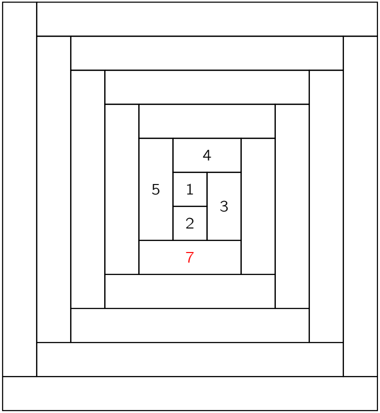
Computing the Fibonacci quilt sequence



All legal sums with at least two elements are at least 6, thus:

$$q_1 = 1,$$
$$q_2 = 2,$$
$$q_3 = 3,$$
$$q_4 = 4,$$
$$q_5 = 5.$$

Computing the Fibonacci quilt sequence

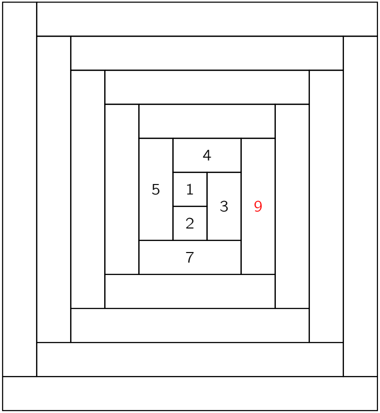


The possible legal sums using $\{1, 2, 3, 4, 5\}$ are:

$$1 = 1,$$
$$2 = 2,$$
$$3 = 3,$$
$$4 = 4,$$
$$5 = 5,$$
$$6 = 2 + 4,$$
$$8 = 3 + 5.$$

Thus, $q_6 = 7$.

Computing the Fibonacci quilt sequence

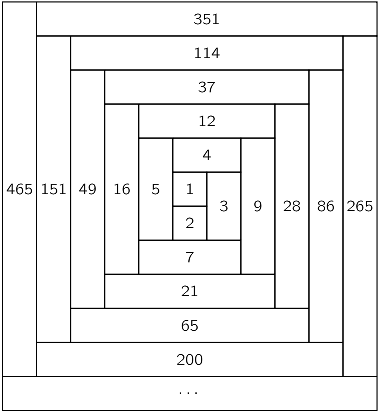


The possible legal sums using $\{1, 2, 3, 4, 5, 7\}$ are:

$1 = 1,$
$2 = 2,$
$3 = 3,$
$4 = 4,$
$5 = 5,$
$6 = 2 + 4,$
$7 = 7,$
$8 = 3 + 5 = 1 + 7,$
$11 = 4 + 7.$

Thus, $q_7 = 9$.

Computing the Fibonacci quilt sequence



And so on...

Recurrence

The behavior of the Fibonacci quilt sequence is well understood:

Proposition (CFHMN, 2020)

Let q_n be the Fibonacci quilt sequence. For $n \geq 5$,

$$q_{n+1} = q_n + q_{n-4}.$$

The triangular quilt sequence

Definition (Triangular quilt sequence; SMALL '22)

Let t_n be the smallest positive integer which **cannot** be written as as a sum of non-adjacent terms in $\{t_1, \ldots, t_{n-1}\}$. Two terms are adjacent if their boxes share an edge in the Padovan spiral.

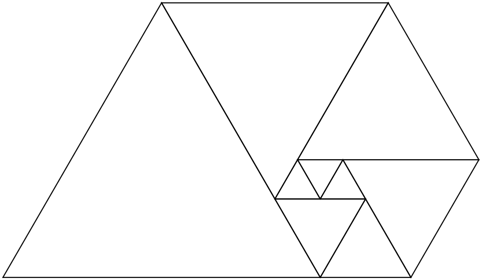


Figure: The Padovan Spiral.

Computing the triangular quilt sequence

We were able to compute the first 50 terms of the triangular quilt sequence.

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	$\cdots$
1	2	3	5	6	11	12	20	23	40	46	80	92	$\cdots$

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Question

Like the Fibonacci quilt sequence, does the triangular quilt sequence follow a recurrence?

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Surprising answer: Based on the first terms, apparently no.

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Surprising answer: Based on the first terms, apparently no.

Bonus fact: This sequence was not listed on the Online Encyclopedia of Integer Sequences.

Half? of a recurrence

Proposition

For $6 \leq n \leq 50$, the equation

$$t_{n+1} = t_n + t_{n-5}.$$

holds if, and only if, $n \in \{6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 31, 33, 35, 37, 39, 41, 43, 46, 48\}$.

Simplifying the quilt sequences

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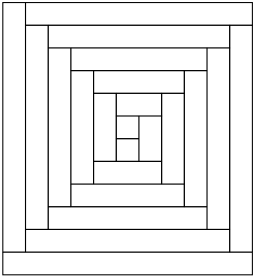
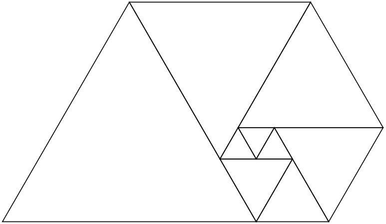


Boxes q_i and q_j are adjacent iff $|i-j| \in \{1,3,4\}$ or $\{i,j\} = \{1,3\}$.



Triangles t_j and t_i are adjacent iff $|i-j| \in \{1,5\}$ or $\{i,j\} = \{1,4\}$.

S-LID decompositions

Fix a set S of positive integers. (For example, $\{1, 3, 4\}$ or $\{1, 5\}$.)

Definition (S-LID decomposition)

An S**-Legal Index Difference (**S**-LID) decomposition** using $\{a_i\}_{i \in I \subseteq \mathbb{Z}_{>0}}$ is a sum of the form

$$N = \sum_{\ell \in L} a_\ell$$

for finite $L \subseteq I$ such that $|\ell_1 - \ell_2| \notin S$ for all $\ell_1, \ell_2 \in L$.

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Example

Let $S = \{2\}$ and $\{a_i\}_{i \in I} = \{a_1, a_2, a_3, a_4\}$. Then

- $a_1 + a_2$ is an S-LID decomposition using $\{a_i\}_{i \in I}$,
- $a_1 + a_2 + a_3$ is not an S-LID decomposition using $\{a_i\}_{i \in I}$ because $|3 - 1| = 2 \in S$.

S-LID sequences

We use S-LID decompositions to construct a sequence.

Definition (S-LID sequence)

The S**-LID sequence** $\{a_i\}_{i=1}^{\infty}$ is defined by:

- a_n is the smallest positive integer that does not have an S-LID decomposition using $\{a_i\}_{i=1}^{n-1}$.

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Example

- The $\{\}$-LID sequence is $\{2^{i-1}\}_{i=1}^{\infty}$,
- The $\{1\}$-LID sequence is the Fibonacci sequence,
- Understanding the $\{1, 3, 4\}$ and $\{1, 5\}$-LID sequences will help us understand the Fibonacci and triangular quilt sequences.

Fundamental lower bound

Fundamental Lower Bound of S-LID Sequences

Let $\{a_i\}_{i=1}^{\infty}$ be the S-LID sequence, and let $k = \max S$. Then, for all $n \geq k+1$,

$$a_{n+1} \geq a_n + a_{n-k}.$$

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$$a_{n+1} \geq a_n + a_{n-k}.$$

To prove it, we show that all numbers smaller than $a_n + a_{n-k}$ have an S-LID decomposition using $a_1, \ldots, a_n$.

Proof of lower bound

- 0 has S-LID decomposition using $a_1, \ldots, a_{n-k-1}$.
- 1 has S-LID decomposition using $a_1, \ldots, a_{n-k-1}$.

$\vdots$

- $a_{n-k} - 1$ has S-LID decomposition using $a_1, \ldots, a_{n-k-1}$.

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We add a_n to each decomposition.
Since indexes n and $n-k-1$ are more than $k = \max S$ apart, we obtain S-LID decompositions.

Proof of lower bound

- a_n has S-LID decomposition using $a_1, \dots, a_n$.
- $a_n + 1$ has S-LID decomposition using $a_1, \dots, a_n$.

$\vdots$

- $a_n + a_{n-k} - 1$ has S-LID decomposition using $a_1, \dots, a_n$.

Proof of lower bound

- numbers $< a_n$ have S-LID decompositions using $a_1, \ldots, a_{n-1}$.
- a_n has S-LID decomposition using $a_1, \ldots, a_n$.
- $a_n + 1$ has S-LID decomposition using $a_1, \ldots, a_n$.

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- $a_n + a_{n-k} - 1$ has S-LID decomposition using $a_1, \ldots, a_n$.

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$\vdots$

- $a_n + a_{n-k} - 1$ has S-LID decomposition using $a_1, \dots, a_n$.

Since a_{n+1} is the smaller number without a S-LID dec. using $a_1, \dots, a_n$, thus

$$a_{n+1} \geq a_n + a_{n-k}.$$

Equality of lower bound ($a_{n+1} \geq a_n + a_{n-k}$)

Example

The $\{1\}$-LID sequence ($k = 1$) is the Fibonacci sequence, which satisfies

$$a_{n+1} = a_n + a_{n-1}.$$

Equality of lower bound ($a_{n+1} \geq a_n + a_{n-k}$)

Example

The first terms of the $\{1, 2, 4\}$-LID sequence ($k = 4$) are

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	$\cdots$
1	2	3	4	6	7	9	12	16	22	29	38	50	$\cdots$

and we have

$$a_{n+1} = a_n + a_{n-4},$$

for all $5 \leq n \leq 50$.

Equality of lower bound ($a_{n+1} \geq a_n + a_{n-k}$)

Example

The first terms of the $\{1,3,4\}$-LID sequence ($k = 4$) are

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	$\cdots$
1	2	3	5	6	8	10	14	19	25	33	44	$\cdots$

and we have

$$a_{n+1} = a_n + a_{n-4},$$

for all $5 \leq n \leq 61$ EXCEPT when $n \in \{5, 7, 11\}$.

Non-equality of lower bound ($a_{n+1} \geq a_n + a_{n-k}$)

Example

The first terms of the {3}-LID sequence ($k = 3$) are

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	$\cdots$
1	2	4	8	9	11	15	23	32	64	79	134	$\cdots$

and we have

$$a_{n+1} > a_n + a_{n-3},$$

for all $4 \leq n \leq 40$ EXCEPT when $n \in \{4, 5, 6, 7, 8, 10, 12, 15\}$.

Weirdness of lower bound ($a_{n+1} \geq a_n + a_{n-k}$)

Example

The first terms of the $\{1,5\}$-LID sequence ($k = 5$) are

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	$\cdots$
1	2	3	5	8	13	14	21	24	43	51	67	105	$\cdots$

For $6 \leq n \leq 47$,

$$a_{n+1} = a_n + a_{n-5}$$

iff $n \in \{6, 8, 10, \quad 21, 23, 25, 27, 29, \quad 34, 36, 38, 40, 42, 44, 46\}$.

Dream False? Theorem

Dream Goal (possibly false?)

The S-LID sequence ($k = \max S$) satisfies

$$a_{n+1} = a_n + a_{n-k}$$

for all sufficiently large n.

Dream False$^?$ Theorem

Dream Goal (possibly false?)

The S-LID sequence ($k = \max S$) satisfies

$$a_{n+1} = a_n + a_{n-k}$$

for all sufficiently large n.

Still, the $\{1, 2, \ldots, k\}$-LID sequence and a couple others satisfy

$$a_{n+1} = a_n + a_{n-k}$$

for all sufficiently large n.

Recurrence Theorem

Theorem (SMALL '22)

Fix any finite set T of positive integers with $c = \max T$. Let $k \gg 0$, and $S = \{1, \ldots, k\} \setminus (k - T)$. Then, the S-LID sequence satisfies

$$a_{n+1} = a_n + a_{n-k}$$

for all $n > k + c$.

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for all $n > k + c$.

Proof is by strong induction on three statements:

$$\begin{aligned} A(n): &\quad a_{n+1} &= &\quad a_n + a_{n-k}, \\ B(n): &\quad a_n + a_{n-k} &> &\quad a_{n-1} + a_{n-1-(k-c)} + a_{n-1-2(k-c)} + \cdots, \\ C_d(n): &\quad a_{n+1} + a_n &> &\quad a_{n+c} + a_{n-d}. \end{aligned}$$

Sketch of the induction step

FLB: $a_{n+1} \geq a_n + a_{n-k}$.
Lemma: $a_n > a_{n-(k-c)} + a_{n-2(k-c)} + \cdots$.
Follows from FLB and requires $k \geq 2(c+1)$.

Step 1: $$B(n) \implies A(n)$$
Uses FLB and definition of a_{n+1}.

Step 2: $$\begin{array}{l} C_d(n-k-1), C_d(n-2k-1), \\ A(n-k-1), A(n-k), A(n-k+c-1) \end{array} \implies C_d(n-k)$$
Manipulation of inequalities.

Step 3: $$C_d(n-k) \implies B(n+1)$$
Uses Lemma and requires $k \geq 2d - 4c + 2$.

Base case of induction

Theorem (SMALL '22)

Suppose that there exists $d > 0$ such that

- $k \geq 2d - 4c + 2$,
- $C_d(n)$ *holds for* $c + d + 1 \leq n \leq k + c + 1 + d$, *and*
- $B(n)$ *holds for* $k + c + 1 \leq n \leq 2k + c + d + 2$.

Then

$$a_{n+1} = a_n + a_{n-k}$$

for all integers $n > k + c$.

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Then

$$a_{n+1} = a_n + a_{n-k}$$

for all integers $n > k + c$.

We proved that these base cases hold for all $k \gg 0$ for fixed T, which finishes the proof for $S = \{1, \ldots, k\} \setminus (k - T)$.

Future work

- Understand the S-LID sequence for more sets S.
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Acknowledgments

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arXiv: 2210.10577
"Recurrence relations for S-legal index difference sequences"
Dantas e Moura, Keisling, Lilly, Mauro, Miller, Phang, and Velazquez Iannuzzelli.
To appear in the Fibonacci Quarterly.